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Agricultural production economics
and resource-use

Agricultural production economics and resource-use

BY

MARTIN UPTON

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Preface

This book is intended as an introduction to the economic principles of production and resource-use. The first part deals with the theory of production under constant prices, approached from both the classical (continuously variable factor proportions) and the modern linear (activity analysis) points of view, together with an introduction to the theory of decision making under risk. The second part is concerned with the use of resources, and is arranged under the usual headings of land, labour, capital, and management. Optimization over space, and location theory are discussed in the chapter on land; and optimization over time in the chapter on capital. Determinants of the size distribution of farms are considered in the chapter on management.

The approach I have used is new, particularly in the use of linear analysis to develop pure theory. A similar approach has been used in some recent texts in pure microeconomic theory but, on the whole, agricultural economists continue to think of linear programming solely as an operational research tool. I believe that linear analysis enriches our theory; for instance in showing that it may be profitable to leave limited resources unused (in disposal) and in illuminating the concept of shadow-prices. Rent on land and return on capital are then seen to be shadow-prices rather than undifferentiated surpluses. Furthermore the complexities caused by introducing time and risk into production theory may be handled more readily in terms of linear relationships.

I have attempted to present the material by means of simple numerical examples and graphs, and have avoided the use of algebra. This constraint has made writing more difficult, but it will have been worthwhile if it makes the material available to those many students who are alarmed by algebraic symbols and formulae. For those who are not so alarmed, the basic ideas are repeated in the mathematical appendices to most chapters, using simple algebra.

The book is not intended as a text in farm management so there is little mention of practical tools such as accounting and budgeting or of practical problems such as taxation and inflation. Neither is it intended as a text in general microeconomics, so there is little discussion of consumer demand, the functioning of markets, general equilibrium, and economic welfare.

I have tried to avoid too much emphasis on British agriculture and would like to think that the theories presented are equally relevant to both tropical and temperate climates and to most economic systems. I have referred throughout to 'the farmer' as the principal decision maker, on the grounds that most of the world's farm production comes from individual enterprises. However the principles of optimization are equally applicable on a communal farm or a state farm.

I hope that the book will prove useful not only to students of agriculture and agricultural economics but also to pure economists interested in the application of theory to a specific industry, to development economists interested in problems of the agricultural sector, and to geographers concerned with spatial aspects of rural land-use.

M. U.

Stratfield Saye
January 1976

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For permission to reproduce Table 1.1 I am grateful to Dr. Boyd, Mr. Dermott, and the *Journal of Agricultural Science*. A part of Chapter 4 is based upon a paper I wrote with Harold Casey for the *Journal of Agricultural Economics*. I am grateful to him and the Agricultural Economics Society for permission to use this material.

Many colleagues at Reading University have helped me by reading parts of earlier drafts and suggesting corrections and improvements. In particular I must mention Ronald Tuck, David Ansell, Jim Burns, Harold Casey and Chris Ritson of the Department of Agricultural Economics and Management; Graham Dalton of the Department of Agriculture; Joan Crampin of the Department of Mathematics; and Ian Wilson of the Department of Applied Statistics. I have benefited greatly from their help.

Contents

PART I: THE ECONOMICS OF FARM PRODUCTION

1. PRODUCTION-RESPONSE CURVES 3
Production and resources—Input–output relationships—Factor–factor relationships—Product combinations—Summary—Mathematical appendix—Suggestions for further reading

2. LINEAR PRODUCTION RELATIONSHIPS 36
Fixed-factor proportions—More than one activity—Constraints and the feasible region—Product–product relationships—Summary—Mathematical appendix—Suggestions for further reading

3. ECONOMIC OPTIMIZATION 73
Economic rationality—The economic optimum for a single variable input—Two variable inputs and the least-cost combination—The cost function and returns to scale—Factor indivisibility—Factor limitations—Costs and the economic optimum—Product supply and factor demand—The return to limited factors—Product exhaustion under constant returns to scale—Alternative products and opportunity costs—The linear programming solution—Summary—Mathematical appendix—Suggestions for further reading

4. DECISION MAKING UNDER RISK AND UNCERTAINTY 130
Variation in agricultural production—Price variation—Probability and expectations—Conditional probability—The fallacy of averages—Risk aversion—Diversification and risk—Choices under risk—Expected utility maximization—Uncertainty and subjective probability—Games against nature—Rules of thumb—Summary—Mathematical appendix—Suggestions for further reading

PART II: RESOURCE-USE

5. NATURAL RESOURCES 195
Land as a factor of production—Population density—Technological change—Soils and climate—Location—Land tenure—Summary—Suggestions for further reading

6. LABOUR
The measurement of labour—The seasonality of farm work—Increasing labour productivity—Income and hours worked—Population growth—The size of the agricultural sector—Summary—Suggestions for further reading 232

7. CAPITAL 260
Basic concepts—Investment—Compounding and discounting—Discounted cash flows—The internal rate of return—Optimization over time—Other replacement problems—The demand for capital—Difficulties in capital theory—Savings and the rate of return—Borrowing and risk—Credit—Multi-

period planning—Summary—Mathematical appendix—Suggestions for further reading

8. MANAGEMENT 321
The managerial function—Entrepreneurship and profit—Managerial ability and farm size—Changing managerial limits—Entrepreneurship and farm growth—Sharing entrepreneurial responsibility—Summary—Suggestions for further reading

APPENDIX: TABLES 341

INDEX 347

Part I: The economics of farm production

1. Production-response curves

Production and resources

Agricultural production depends upon the natural growth and reproduction of plants and animals, but farmers take control of these processes and stimulate them to yield food and other commodities for human consumption. For this activity of production farmers need a supply of productive resources such as land, seeds, breeding stock, human skill and effort, tools and machines. These productive resources, also known as factors of production, are usefully grouped into four main categories: (1) natural resources; (2) labour; (3) capital; (4) management.

The first category consists of those gifts of nature which are not the result of human effort, and include land, water, and local climate. 'Land' in this sense is defined as the 'original and indestructible powers of the soil',[1] but it must be admitted that the distinction between natural resources and the other categories is arbitrary. For land has often been made productive as a result of human effort in clearing, cultivating, manuring, and perhaps irrigating and draining; the supply of water has often been improved by the building of dams or storage tanks or the construction of canals. These results of past efforts are a form of capital and should be included in the third category.

The term 'labour' describes the effort of human beings, including that of farmers, their families, and the hired workers. The effort may be either manual or mental, skilled or unskilled, and the total amount depends upon both the number of persons available and the amount of effort each one is prepared to make. Labour can become more productive as a result of time and effort devoted to training and education. Here again the skill and knowledge acquired may be classified as capital.

'Capital' represents resources which are the result of past human effort. This category includes a wide diversity of items ranging from durable capital such as buildings, dams, roads, and machinery to stocks of materials like seed or fertilizer which may be used up within a single season. Capital is often confused with money savings in everyday speech, but for our purposes it is important to distinguish between them for

[1] Ricardo, D. (1817). *The principles of political economy and taxation*, p. 33. Everyman Edition, Dent, London.

money is not itself a productive resource: it becomes productive only when it is used to buy or hire physical items of capital.

These resources of land, labour, and capital are organized into productive units which, in agriculture, are called farms. The layman tends to think of a farm as a piece of land with its associated buildings, and therefore that an individual may manage or operate more than one farm. However, from an economic point of view a farm is best defined as all the resources under the control of one individual or group of individuals who take joint decisions. In this sense a farm may be made up of several separate blocks of land.

'Management' describes the function of taking decisions about how land, labour, and capital resources should be used and also of carrying out these decisions. All production implies the taking of some risk, since decisions are made and resources committed on the basis of expected yields and prices. Actual outcomes may be worse than hoped for because of either bad luck or bad decisions. In some forms of business, risk taking may be treated as a separate, and fifth, category of productive resources but in farming the farmer himself takes the risks and bears the consequences of his own decisions.

Input–output relationships

The growth and reproduction of plants and animals, like most other productive processes, takes time. Hence, the use of productive resources and the yield of farm products must be related to a particular time interval. The number of pigs produced per sow may be increased by shortening the farrowing interval as well as by increasing the number of pigs per litter. Arable crop production may be increased by shortening the length of fallows or leys as well as by increasing yields. For an individual crop- or livestock-production process the appropriate time interval is a single production cycle but, in order to compare alternatives, a standard unit of time must be used—usually the calendar year.

Some productive resources such as stocks of seeds, fertilizers, or feeding-stuffs actually get used up within a single production period whereas others such as land, labour, machinery, and buildings merely provide services which contribute to the productive process. The quantities of productive resources, or the services they provide, which are used on a farm over a given production period are called inputs to distinguish them from outputs, which are produced.

In general it is impossible to list all the inputs involved in producing a particular crop or livestock product. Indeed some 'inputs' such as sunshine, rainfall, pests, and diseases are not completely under the farmer's

control. These variable features of the environment are in part responsible for the variation in farm output and hence the risks farmers face. For some purposes it is necessary to consider very specific inputs such as the quantity of nitrogenous fertilizer, the amount of labour with particular skills at a specific period of the year or the area of land of a precise soil type. For other purposes inputs may be aggregated into the broad groups of land, labour, and capital (although there are serious difficulties in measuring aggregate capital).

It is generally assumed that certain inputs are unavoidably fixed, at least over a single production period. For instance, a farmer may not be able to vary the amount of land he farms or the area of tree crops he grows from one year to another. Generally speaking, land, labour, machinery, and buildings provide a flow of services which cannot readily be varied in total, whereas the amount that is used from stocks of seeds, fertilizers, or feeding-stuffs can be varied quickly and easily. Given a sufficiently long period of time most inputs can be varied but the managerial ability of the farmer himself may set a limit on the quantities of other resources controlled.

The level of output achieved in a given period of time is dependent upon the quantities of inputs; there is a technical relationship between inputs and outputs. As we have suggested, there is some uncertainty about the precise level of output which will be obtained because of the unpredictability of weather and pests and diseases. However, the uncertainty is not so great as to prevent scientists from estimating the production relationships or to prevent farmers from using their knowledge of those relationships in reaching management decisions.

The input–output relationships are determined by the techniques of production. Indeed they define the technology available to the farmer. They are altered by technological change. Together with the levels of fixed inputs, these relationships determine what it is feasible for the farmer to produce. They set limits on the alternatives open to him. Every farmer therefore needs to know something of the way in which his resource inputs influence his output.

The biological processes of growth, development, and reproduction are very complex, so the relationships between inputs and outputs are also likely to be complex and subject to some uncontrollable variations. For management decision making, simplified theories, or models, of production response are needed. These models cannot conform exactly to reality. If they could, they would apply only to individual very specific sets of circumstances. The simpler models are more useful because they can be applied more generally and because they are more readily

understood by farmers and their advisors. However, increased generality and flexibility is obtained at the cost of detail and realism.

Two alternative models of production response will be discussed in this book. Both models have been subjected to some testing against observed facts and have proved to be sufficiently realistic for use in guiding some farm management decisions.

This chapter deals with the model of smooth curved functions relating factor inputs and product outputs, while the next chapter will be concerned with straight-line relationships and fixed constraints.

The law of diminishing returns

The results of many experiments on the influence of fertilizers and irrigation water on crop yields and the effects of feeding rates on livestock production are consistent with the assumption of smooth, continuous relationships between inputs and outputs.[2] Such relationships are called production functions.[3] The important characteristics of most production functions are illustrated by the results of fertilizer trials on potatoes given in Table 1.1. These results show the effects of four different levels of nitrogen application in combination with three different levels of phosphate and potash on potato yields measured in tonnes per hectare.

The amount by which the yield, or total product, is changed by a single unit change in nitrogen input is called the 'marginal product' per unit of nitrogen. Marginal products are shown in the third, fifth, and seventh columns of the table. In each of these columns the marginal products are seen to diminish with increasing nitrogen applications. Indeed, for zero levels of phosphate and potash the marginal product of the second unit of nitrogen is negative and of the third unit of nitrogen is even more negative. In fact it is unlikely that any farmer will be interested in fertilizer

[2] For some empirical examples see Heady, E. O. and Dillon, J. L. (1961). *Agricultural production functions.* Iowa State University Press, Ames.

Note however that many agricultural scientists now take the view that such relationships are better represented as being linear but subject to limiting factors or constraints. For examples see Boyd, D. A. (1972). Some recent ideas on fertilizer response curves. *Proceedings of the 9th International Congress of the Potash Institute*, pp. 461–473. Institut International de la potasse, Berne, or Fisher, C., Morris, T. R., and Jennings, R. C. (1973). A model for the description and prediction of the response of laying hens to amino acid intake. *Br. Poul. Sci.* **13**, 469–84.

This model of linear production response will be discussed in Chapter 2. However there are still enough supporters of the 'smooth, continuous-curve theory' to justify further discussion in this chapter.

[3] The term 'production function' may be used to describe any numerical estimate of the relationship between inputs and outputs, such as a simple enumeration or the model described in the next chapter. However the term is often used to mean more specifically the smooth, continuous type of relationship.

combinations for which marginal products are negative since this implies that applications of fertilizers are so high as to actually depress yields. For the combinations which are of interest to farmers, marginal products are all positive but decreasing. Because the fertilizer is applied to a fixed area of land and a fixed quantity of seed and other inputs it becomes more and more difficult to increase yields and hence extra nitrogen becomes less and less effective as rates of application are increased.

This case of diminishing returns to fertilizer in potato production is an example of a general rule, the so-called 'law of diminishing returns',

TABLE 1.1

Potato response to fertilizers on fen soils
Yields: tonnes per hectare

Level of nitrogen	Levels of phosphate and potash					
	00		11		22	
	Total product	Marginal product	Total product	Marginal product	Total product	Marginal product
0	26·9		28·4		—	
		6·5		6·2		—
1	33·4		34·6		32·6	
		−0·8		3·8		6·3
2	32·6		38·4		38·9	
		−3·2		−3·0		0·5
3	29·4		35·4		39·4	

After Boyd, D. A. and Dermott, W. (1964). Fertilizer experiments on main crop potatoes. *J. Agric. Sci.* **63**, 249. Also see Cooke, G. W. (1967). *The control of soil fertility*. Crosby Lockwood, London.

which may be stated as follows. If the inputs of one factor are increased by successive equal amounts, the rates of use of all other factors being held constant, then the marginal product per unit of the variable factor will eventually decline.

The other important feature of production functions which is apparent from this table is the positive interaction between inputs. Thus an increase in inputs of phosphate and potash generally brings about an increase not only in total product but also in the marginal product of nitrogen for all units of nitrogen after the first. The extra product obtained by increasing inputs of phosphate/potash and nitrogen together is greater than the sum of the marginal products of these two inputs. For instance, with zero level of phosphate/potash, the marginal product of the second unit of

nitrogen is −0·8 tonnes per hectare. By comparing the first and third columns we see that the marginal product of 1 unit of phosphate/potash is 34·6 − 33·4 = +1·2 tonnes per hectare. The sum of these marginal products is +1·2 − 0·8 = 0·4 tonnes per hectare but the actual increase in total product when both inputs are increased together by 1 unit is 38·4 − 33·4 = 5·0 tonnes per hectare.

In practice the total yield of potatoes would depend on many other factors, including the timing and method of fertilizer application as well as the amounts applied. For other factors of production, such as labour, the true relationship between inputs and outputs may be far more complex. Nevertheless, useful insights may be gained by assuming similar smooth, continuous relationships between all inputs and outputs.

Response to a single variable input

In order to study the relationship between a single variable input and the product output, sometimes called the factor–product relationship, we assume initially that inputs of all resources except one are fixed. The factor–product relationship may be expressed in terms of total product, average product, marginal product, or elasticity of response as shown for a hypothetical example in Table 1.2. The elasticity of response, shown in the last column of this table, is the relative change in output divided by the relative change in input. It is most conveniently thought of as the percentage change in output resulting from a 1 per cent increase in input and is a pure number, independent of the units in which inputs and outputs are measured. Alternatively, it may be calculated as the marginal product divided by the average product (see notes at the end of this chapter).

The total product is always the sum of the marginal products of the units of variable resource used.[4] For example, in Table 1.2 the total product from 3 units of labour, namely 31 bags of grain, is the sum of the marginal products 7 + 13 + 11 bags. This relationship follows from the definition of marginal product as the addition to total product. Initially, for the first two units of labour input, marginal product per unit increases—a situation which may arise because these units of labour are too thinly spread over the fixed quantity of land and other resources to grow and harvest the crop properly. Further increases in labour-use, beyond 2 units, are associated with diminishing marginal products which eventually are negative for the sixth and seventh units of labour.

If we assume that labour inputs and grain outputs can be varied continuously, we can illustrate the effect of varying inputs on total, average,

[4] Plus the amount of product, if any, obtained at zero input.

and marginal products by means of smooth curves as shown in Fig. 1.1. The total product curve is sometimes called 'the response curve'. The marginal product may now be estimated at any point on the response curve as the slope of the curve at that point. It is equal to the slope of a tangent (a straight line which just touches without crossing) to the curve at that point, if such a tangent can be drawn. Two important points may be distinguished on the response curve. The first of these occurs at a labour input of 2·6 units where the average product, also known as the

TABLE 1.2

The factor–product relationship (hypothetical data)

Labour used per year (units)	Total physical product per year	Average product	Marginal product	Elasticity of response
	(Bags of grain)			
0	0	0		
			7	
1	7	7·0		
			13	1·9
2	20	10·0		
			11	1·1
3	31	10·3		
			6	0·6
4	37	9·3		
			3	0·3
5	40	8·0		
			−1	−0·1
6	39	6·5		
			−4	−0·6
7	35	5·0		

Average product $= \dfrac{\text{Total physical product per year}}{\text{Units of labour used per year}}$

Marginal product = Increase in total product; i.e. difference between successive figures in total physical product per year column.

Elasticity of response $= \dfrac{\text{Marginal product}}{\text{Average product}}$

productivity, of labour is at a maximum. At this point marginal product is equal to average product and elasticity of response has a value of exactly one. If land and other fixed inputs were available free of cost and the aim was to maximize output per unit of labout this point would be chosen. The second important point occurs at a labour input of 5·3 units where total product is at a maximum. At this point the marginal product per unit of labour is zero and so too is the elasticity of response. This

point is sometimes called the technical optimum since the average product per unit of fixed resources is at a maximum.

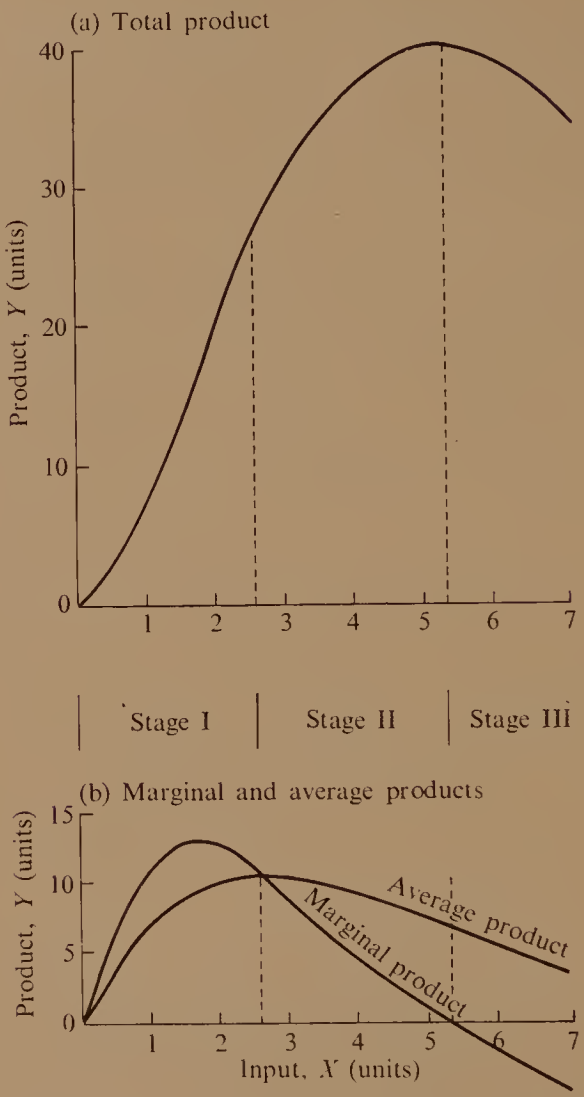


FIG. 1.1. (a) Hypothetical response curve. This shows how the rate of input of a single variable factor influences the rate of output, when all other factors are held constant. (b) The marginal product curve shows the *extra* output obtained per unit of input. It is larger than the average product where this is rising, in stage I, but smaller than the average product where this is falling, in stages II and III. In stage III marginal product is negative and total product or output is falling.

These two points divide the response curve into three stages, the first and third of which are often eliminated from analysis of farm-management decisions as being irrational areas of production. In the first

stage where labour inputs are below the point of maximum average product, elasticity of response is greater than one and the marginal product is greater than the average product. Each extra unit of labour adds more to total product than each of the units already employed so the average product must be increased by employing more labour. In this stage increased labour use causes increases in the productivity of labour and all other inputs as well. Hence, no matter how inputs and outputs are valued, so long as they have some positive value, either the end of stage I at 2·6 units of labour input, or the beginning of stage I at zero units of labour input and zero output, must be preferable to any level of labour input *within* stage I. It is irrational to choose a level of labour within stage I.

Similarly it is irrational to choose a level of input in stage III, beyond the level of maximum total product. Here the marginal product per unit of labour is negative: extra units of labour actually decrease the total product, so the productivity of labour and all other inputs is falling at this stage.

We usually find the marginal and average products are positive yet diminishing in practice because most farmers operate in stage II and avoid the irrational areas. In this second stage, elasticity of response is less than one, marginal product is less than average product, and both marginal and average product per unit of variable factor decreases as inputs are increased. This means that the response curve bulges upwards in stage II.

It is quite possible that an inefficient farmer would produce less output for a given level of input than is given by the function. For instance he may produce only 29 bags of grain by using 3 units of labour. Thus the response curve represents the maximum possible level of output for a given level of input. In this sense it separates inefficient levels of output from those above the curve which are infeasible. In stage II, the response curve bulges upwards, and it is said to be concave from below. This means that a straight line joining two points on the curve cannot pass outside the curve. Clearly the response curve in stage I is not concave, because a straight line joining the point zero input, zero output to the point 2 units of labour, 20 bags of grain output would pass above the response curve.

Response to two inputs

We now consider the effects of varying inputs of two factors of production. Again, hypothetical figures are used to illustrate the relationships involved. The total physical product in bags of grain for different

combinations of two factors, labour and land, are given in Table 1.3. These figures therefore represent the production function relating output to inputs of labour *and* land. Each row in this table represents the response curve for labour at a given level of use of the other resource land. It will be noticed that the data used in the top row for 1 unit of land are the same as those given in Table 1.2. At any point in Table 1.3 the marginal product of labour is the difference between two consecutive values in the same row. Each column in the table represents the response curve for varying inputs of land with a given level of labour-use. The marginal product per unit of land is the difference between two consecutive values in the same column.

TABLE 1.3

Hypothetical production function data relating amount of output in bags of grain (Y) to varying combinations of labour (X_1) and land (X_2)

	Labour input, X_1 (units)							
Land input, X_2 (units)	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	7	20	31	37	40	39	35
2	18	33	45	52	57	57	54
3	27	42	55	62	65	67	65
4	33	48	61	68	71	73	72
5	37	53	66	73	76	78	77
6	39	58	69	76	79	81	80
7	40	60	70	77	80	82	81

If we assume that inputs of land and labour can be varied continuously, smooth curves can be fitted to the data given in each row or each column. In this case the data represent points on a smooth surface in three dimensions as drawn in Fig. 1.2. This diagram shows the interlaced single-variable response curves crossing the surface. Of course there are an infinite number of such single-variable response curves when each input is continuously variable but for illustrative purposes they are shown at one unit intervals.

We can see from Table 1.3 and Fig. 1.2 that, as in the earlier example of potato response to fertilizers, there is an interaction between the two variable inputs. This is even more obvious if the separate labour-response curves for different levels of land-use are drawn on the same two-dimensional diagram as in Fig. 1.3.

The interaction means that if inputs of one factor are increased in relation to inputs of another factor, although the marginal product per unit of the first factor eventually declines, the marginal product of the

second factor rises.[5] In our example the marginal product of each unit of labour is higher when it is used with two units of land than when it is used with one. For instance the fourth unit of labour on 1 unit of land has a marginal product of 6 bags of grain but on 2 units of land it has a marginal product of 7 bags of grain. Similarly the marginal product of the second unit of land is increased from 11 bags up to 19 bags of grain

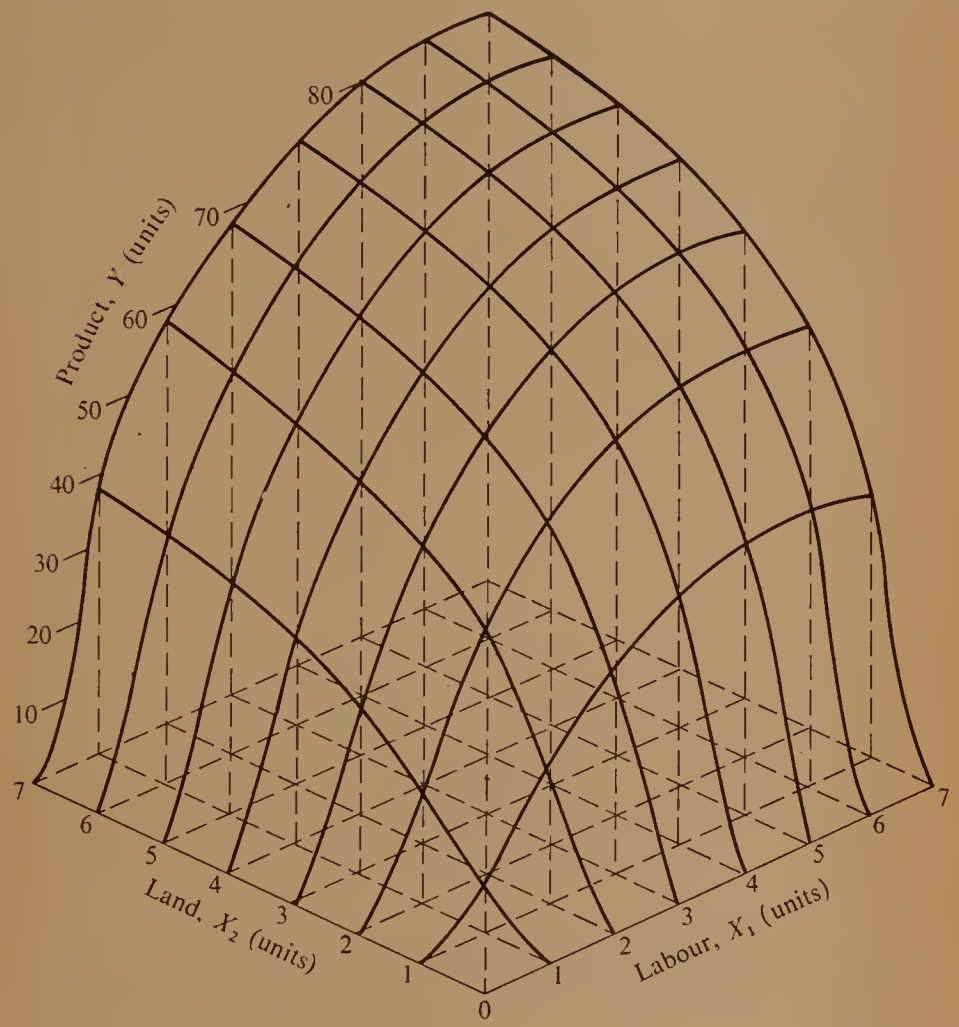


FIG. 1.2. Response surface for two inputs with interlaced single variable-response curves. Individual single-variable response curves for labour are represented by vertical slices parallel to the labour (X_1) axis. Likewise response curves for land are parallel to the X_2 axis.

[5] Negative interactions are possible where factors of production are close substitutes. Then, as inputs of one factor are increased in relation to the inputs of the close substitute, the marginal product per unit of the latter may fall. However, in general we may expect positive interactions to dominate.

as the labour input increases from 1 up to 7 units. In general the marginal product per unit of one resource increases as the quantities of other resources used with it are increased. For instance the marginal product

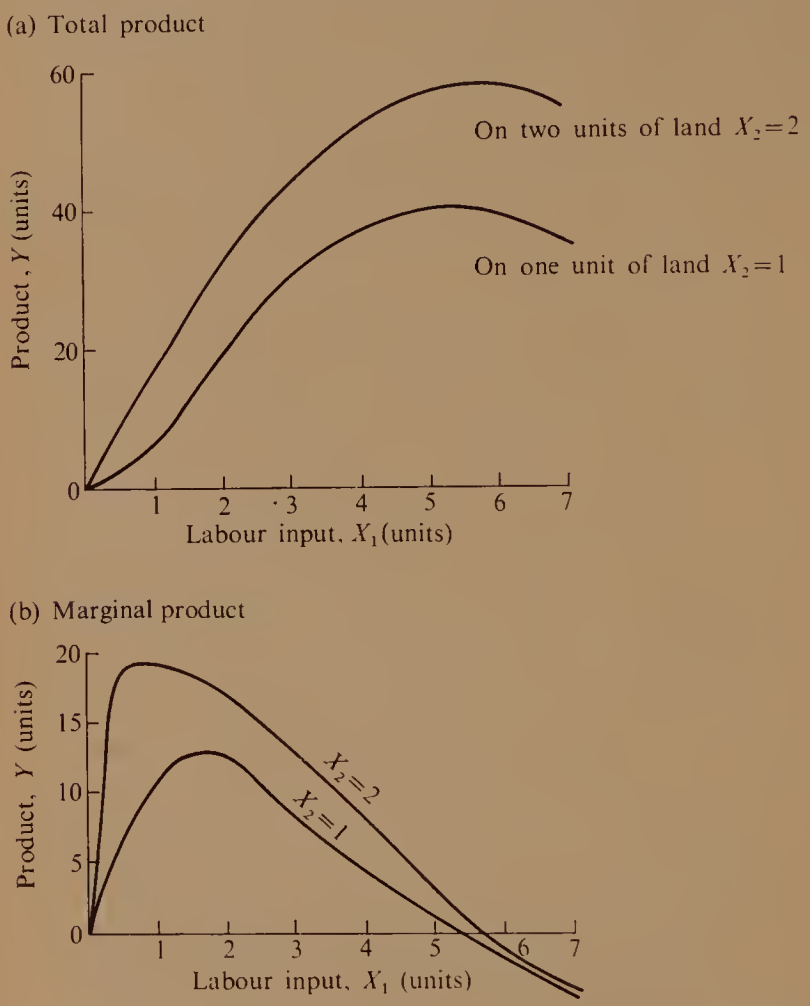


FIG. 1.3. The influence of land input on the labour response curve. Total average and marginal products of labour are increased by using more land.

per hectare of land tends to increase with increasing population density, and hence labour-use per hectare. Also the marginal product per unit of a variable factor depends upon the level at which limiting factors are fixed. Thus experiments carried out on research stations with a high level of management may give responses to variable inputs which are much higher than those obtained on farms with lower levels of management.

Factor–factor relationships

The relationship between inputs of two variable factors of production is best examined by considering the alternative combinations which would produce a given level of output. Using the example data we will consider the combinations of land and labour which will yield an output

TABLE 1.4

The factor–factor relationship

Labour, X_1 (units)	Land, X_2 (units)	Rate of technical substitution: units of land replaced by 1 unit of labour: Decrease in X_2/Increase in X_1
(a) For 40 bags grain output		
1	7·0	
		4·2
2	2·8	
		1·0
3	1·8	
		0·5
4	1·3	
		0·3
5	1·0	
		−0·1
6	1·1	
		−0·2
7	1·3	
(b) For 50 bags grain output		
1		
2	4·2	
		1·6
3	2·6	
		0·8
4	1·8	
		0·3
5	1·5	
		−0·2
6	1·7	
		−0·2
7	1·9	

of 40 bags of grain. From Table 1.3 it will be apparent that this level of output could be produced by a combination of 5 units of labour and 1 unit of land *or* by a combination of 1 unit of labour and 7 units of land. However there are other combinations not specified precisely in this table which would also produce 40 bags of grain: for instance 2 units of labour and between 2 and 3 units of land. These intermediate combinations are

given in Table 1.4. Since we have assumed that inputs of labour and land can be varied continuously, these data represent points on a smooth curve, joining all combinations of land and labour which give equal levels of output. The curve is therefore called an iso-product curve, the term 'iso' meaning equal. It may also be known as an iso-quantity curve or

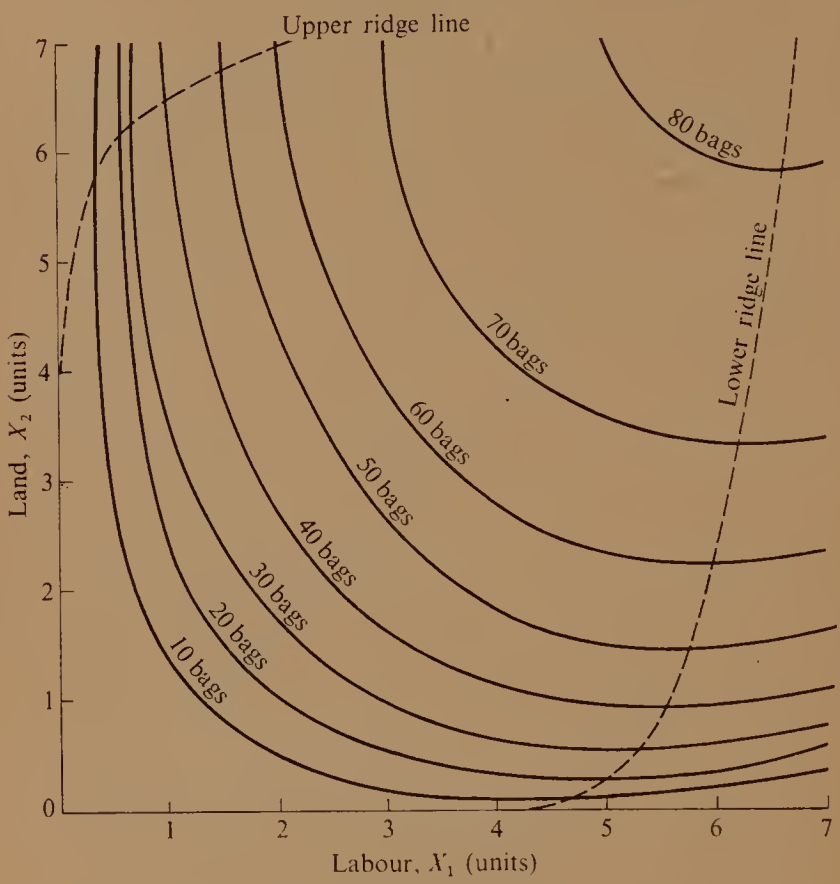


FIG. 1.4. Isoquants for land and labour in grain production. Isoquants join all possible combinations of land and labour which will produce a given level of output. They normally have a negative slope between the ridge lines and are convex to the origin.

'isoquant' for short. In part (b) of Table 1.4 the data represent points on the 50-bag isoquant.

These isoquants are plotted on a graph relating inputs of land to inputs of labour in Fig. 1.4. This graph is similar to a contour map used in laying out terraces or in plotting the topography of an area of land. However whereas contour lines on maps represent heights, isoquants are product

contours. The isoquant graph is really a plan view of the three-dimensional diagram which is repeated in Fig. 1.5 with the isoquants drawn in. There are an infinite number of such isoquants when output is continuously variable, but we will concentrate on those at 10-bag intervals.

Returning to the two-dimensional factor–factor relationship represented in Table 1.4 and Fig. 1.4, the following characteristics should be

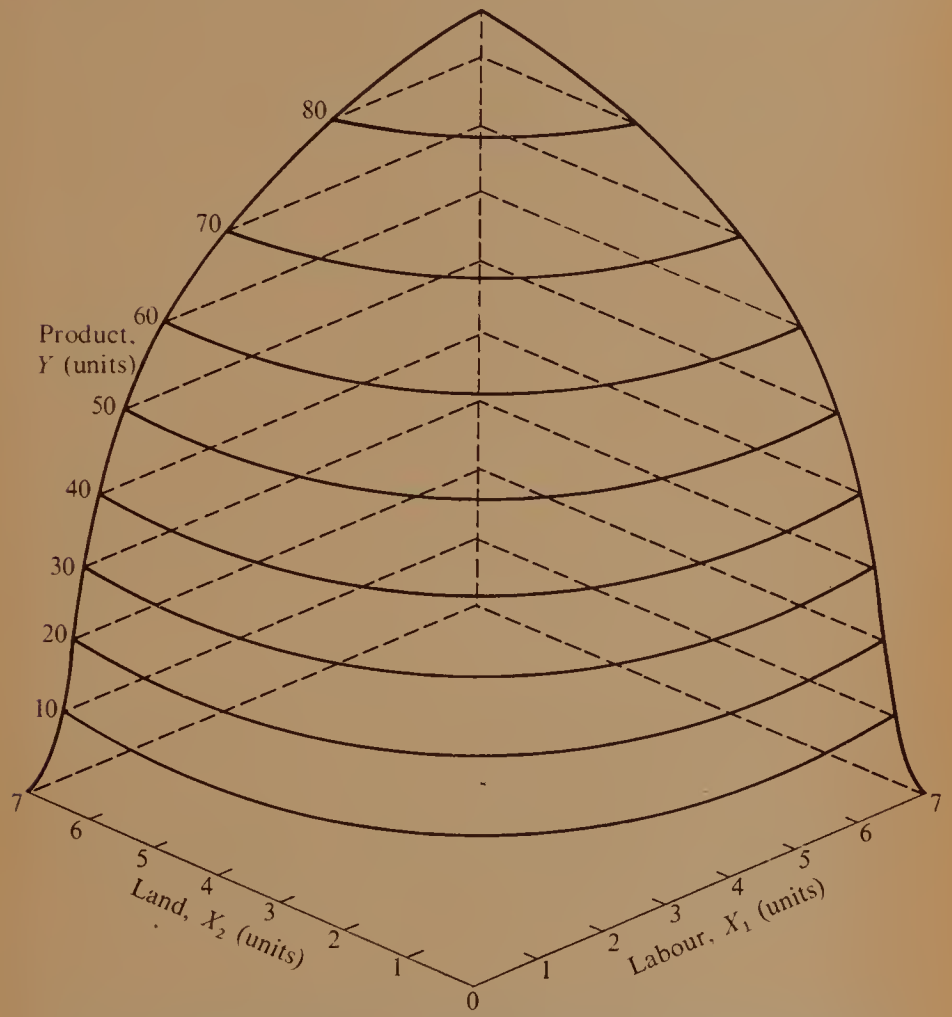


FIG. 1.5. Product contours on the response surface. The isoquants represent horizontal slices through the surface at specific levels of output.

noted, as they apply to most isoquants for any pair of factors. First, the slope of the curve shows the quantity of land replaced by one extra unit of labour. It therefore represents the rate of technical substitution.[6] The

[6] It is called the rate of *technical* substitution to distinguish it from the rate of value substitution which depends upon consumer tastes. It is frequently referred to as the *marginal* rate of substitution.

calculated rate of substitution of labour for land is given in the third column of Table 1.4. It is obtained simply as the difference between successive values in the second column of the table.

The second point to note is that isoquants generally have a negative slope, so strictly speaking the slope of the curve is the negative of the rate of technical substitution. This is only to be expected. If we use more of one resource we would hope to use less of the other resource in producing a given level of output. It can be shown (see appendix to this chapter) that the rate of technical substitution of labour for land is equal to the marginal product of labour divided by the marginal product of land. The smaller is the marginal product for land, the larger is the amount of land that can be substituted for by one unit of labour. In fact there may be areas where the isoquants slope upwards, as is the case for the 40-bag isoquant for labour inputs of over 5·5 units, but these are irrational areas of production since the marginal product of one or other resource is negative. The inputs are no longer substitutes for each other. A farmer who uses 7 units of labour and 1·3 units of land to produce 40 bags of grain could obtain just as much output using less of both factors. In fact we may add 'ridge lines' (the broken lines in Fig. 1.4) which separate the areas where the rate of technical substitution is positive and the slope of the isoquant negative, from the irrational areas. On the lower ridge line the marginal product of labour is zero whereas on the upper ridge line the marginal product of land is zero.

The third point is that isoquants for higher levels of output will normally lie above and to the right of isoquants for lower levels of output. In Fig. 1.4, the 50-bag isoquant lies above and to the right of the 40-bag isoquant. This simply means that it will normally require more of either or both resources to produce more output. For similar reasons no isoquants intersect. We would not expect a given combination of land and labour to produce both 40 bags of grain and 50 bags of grain.

The fourth point to notice is that the curves are convex to the origin. This means that the rate of technical substitution diminishes as more of one factor is used to replace the other. In our example each added unit of labour substitutes for, or replaces, less land than the previous unit.

The reason for diminishing rates of substitution is that one factor is rarely a *perfect* substitute for the other. As we use less land and more labour, increasing amounts of labour are needed to make up for each unit less land used. This is merely an expression of the law of diminishing returns. As more labour is used its marginal product diminishes so larger amounts of labour are needed to substitute for one unit of land.

Although land and labour may not be perfect substitutes, some inputs

are more nearly so. For instance horses and tractors may be nearly perfect substitutes as sources of motive power. The isoquants for perfect substitutes are straight lines as shown in Fig. 1.6 implying a constant rate of technical substitution. A given level of output may be obtained with either horses alone, or tractors alone, or some combination of both.

Finally, the spacing of the isoquants along any ray from the origin tells

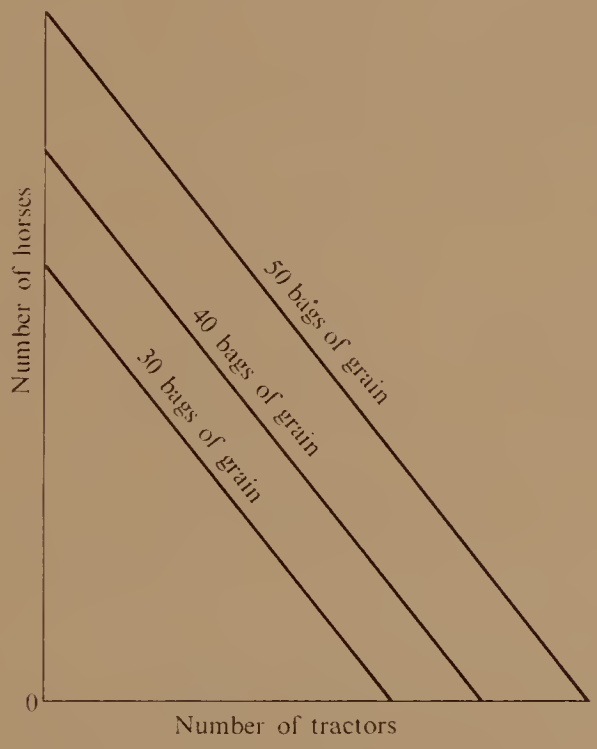


FIG. 1.6. Isoquants for perfect substitutes. If horses and tractors were perfect substitutes the isoquants would be straight lines cutting both axes. This implies that any given grain output can be achieved with either horses or tractors alone.

us the effect of increasing inputs of both factors in constant proportions: that is increasing inputs to scale. If isoquants are equally spaced this means that there are constant returns as the two inputs are increased together. Where, as in the present example for outputs up to 20 bags, the isoquants get closer together, there are increasing returns to both factors. Conversely where isoquants become further apart, as in the present example for outputs over 20 bags, there are decreasing returns to increases in both inputs.

Returns to scale

Strictly speaking, the law of diminishing returns applies only where the inputs of *one* factor are increased, the quantities of *all* other factors being held constant. When inputs of two or more variable factors are increased together, the marginal physical products are unlikely to diminish as rapidly as when a single resource is increased alone. For instance the marginal product of labour will diminish less rapidly if each extra unit of labour has additional land to work with. However, even when two or more factors are increased together, returns are likely to diminish eventually so long as at least one other factor is fixed.

'Returns to scale' are obtained when *all* inputs are varied in the same proportion. If output increases by the same proportion, returns to scale are constant for the range of input combinations under consideration. They are increasing if output increases by a greater proportion and decreasing if it increases by a smaller proportion. Returns to scale may be estimated by adding up the elasticities of response for individual factors of production. For instance, if the elasticity for land is 0·2 and for labour 0·6 then if inputs of both land and labour are increased by 1 per cent, output will increase by 0·2 per cent plus 0·6 per cent, which is 0·8 per cent. Similarly the effect of increasing *all inputs* by 1 per cent (that is, increasing scale by 1 per cent) is estimated by the sum of all the elasticities of response. This sum of elasticities of response is sometimes known as the scale coefficient. If the scale coefficient is equal to one, it means that a 1 per cent increase in scale leads to a 1 per cent increase in output: there are constant returns to scale. If the scale coefficient is less than one there are decreasing returns to scale and if the scale coefficient is greater than one there are increasing returns to scale. The scale coefficient may vary over different parts of the production function. It is sometimes argued that a stage of increasing returns is followed by a stage of constant returns and finally decreasing returns are met with as scale is increased.

However, more careful thought suggests that if all inputs could be increased in strict proportion then constant returns to scale *must* be obtained. After all, it is logical to expect 200 cows to produce just twice as much milk as 100 cows, provided all the other inputs were doubled also. Likewise we might expect 100 hectares of wheat to yield 10 times as much grain as 10 hectares if all inputs were increased in proportion. Only because rarely, if ever, can *all* inputs be varied in proportion are increasing and decreasing returns met with in practice.

The assumption of constant returns to scale is convenient in that if activities are expanded to scale, the input–output relationships are

straight lines. It means that for a given combination of inputs the marginal product of each input is constant regardless of the scale. Thus if all inputs are increased to scale and there are constant returns, the marginal product of the tenth man-hour of labour per hectare of wheat land is exactly the same on 10 hectares of wheat as on 100 hectares of wheat. This theory of straight-line relationships will be developed in the next chapter. Note that the law of diminishing returns still applies if the factor proportions are allowed to vary. Indeed the law is sometimes known as 'the law of variable proportions'. It is only when the factor proportions are held constant that production is expanded to scale and marginal products may remain constant if there are constant returns. In the absence of constant returns to scale, however, marginal products may vary with scale, even if factor proportions remain constant.

Product combinations

Most farmers are faced with a range of alternative crops they could grow and livestock they could keep. The different sections of the farm, each devoted to the production of one kind of crop or animal, are known as enterprises. In theory, if a farmer had access to unlimited resources, enterprises would be independent of each other; an increase in one would neither aid nor hinder another. There would be a separate production function for each enterprise.

However, this is a rather unrealistic view as there are almost always some resources which are limited to a fixed amount. In such circumstances, enterprises will compete with each other for some of these fixed resources so that expansion of one enterprise must be accompanied by a reduction in another. This gives rise to so called product–product relationships. It should be noted that enterprises do not necessarily compete for *all* fixed resources. Even where the total amount of land available is fixed, intensive pig or poultry production need not compete with other enterprises for land. Specialized machines and equipment may be specific to a single enterprise: a Danish piggery or a potato harvester, for instance. Thus enterprises compete for some fixed resources but not for others.

Some enterprises yield more than one product, for instance milk and calves, or mutton and wool. If such joint products are obtained in fixed proportions, then for purposes of analysis the joint products may be treated as a single product, the output of which is related to inputs through a single production function. Where there is scope for varying the proportions in which the two products are obtained they may be

treated as two competing alternatives, as in the case of competing enterprises.

A single limiting factor and two products

The product–product relationship is best illustrated in the first instance by a hypothetical example in which just two alternatives are considered, grain production and root-crop production. These enterprises (products) compete for the farm labour resources which amount to 7 units in total. It is assumed that other resources specific to each enterprise are fixed so that as labour inputs are increased, the marginal returns diminish.[7] Hypothetical total and marginal products for labour in each of the two enterprises are given in Table 1.5.

TABLE 1.5

Labour productivity in two alternative enterprises

Labour used (units)	Total physical product in grain production (bags)	Marginal physical product in grain production (bags)	Total physical product in root crop (tonnes)	Marginal physical product in root crop (tonnes)
0	0		0	
		7		6
1	7		6	
		13		6
2	20		12	
		11		6
3	31		18	
		6		6
4	37		24	
		3		5
5	40		29	
		−1		1
6	39		30	
		−4		0
7	35		30	

From this table we can see that if all 7 units of labour are used in root-crop production the total product is 30 tonnes of roots. No labour is available for grain production so the output of grain is zero. Now if 1 unit of labour is transferred from the root-crop to grain production, there will still be an output of 30 tonnes of roots but in addition 7 bags

[7] Although the total labour available is *fixed* at 7 units, the quantity used in either enterprise is *variable* between the limits of 0 and 7 units. Hence the law of diminishing returns will apply.

of grain will be produced. Presumably this combination of outputs will be preferable to 30 tonnes of roots alone so it would be irrational to use all 7 units of labour in root-crop production. However, if a second unit of labour is transferred to grain production so that only 5 units are used in root production this second unit has a 'cost' in terms of a reduction in root output from 30 tonnes to 29 tonnes. This cost measured in terms of alternative opportunities forgone, is known as the 'opportunity cost'. In this case the opportunity cost of increasing grain output from 7 bags to

TABLE 1.6

The product–product relationship

Labour used in root production, X (units)	Units of labour used in grain production $7-X$ (units)	Total physical product roots (tonnes)	Total physical product grain (bags)	Rate of product transformation (bags of grain per tonne of roots)
0	7	0	35	
				−0·7
1	6	6	39	
				−0·2
2	5	12	40	
				0·5
3	4	18	37	
				1·0
4	3	24	31	
				2·2
5	2	29	20	
				13·0
6	1	30	7	
				∞
7	0	30	0	

20 bags by transferring 1 unit of labour out of root production is 1 tonne of roots. It will be noted that as more labour is transferred from roots to grain production the marginal product per unit of labour in grain production diminishes. At the same time, because there are diminishing returns to labour in root production, when we reduce the amount of labour used in this crop, the output lost per unit of labour removed must increase. Thus the opportunity cost of grain production increases.

Similarly, if labour is transferred out of grain production into the root-crop enterprise, the opportunity cost, in terms of grain forgone, will increase.

We may tabulate all possible combinations of grain and root-crop outputs, for a fixed level of 7 units of labour, to illustrate the product–product relationship as in Table 1.6. The first four columns of this table are obtained directly from Table 1.5.

The quantities of roots and grain given in columns 3 and 4 which can be produced with 7 units of labour represent points on the 'production-possibility curve'. If labour inputs on each enterprise can be varied continuously then the production-possibility curve will appear as a smooth curve as shown on a graph in Fig. 1.7. This curve represents the maximum production possibilities for a given resource level. It is therefore sometimes known as a 'production-possibility frontier', 'boundary', or 'envelope'.

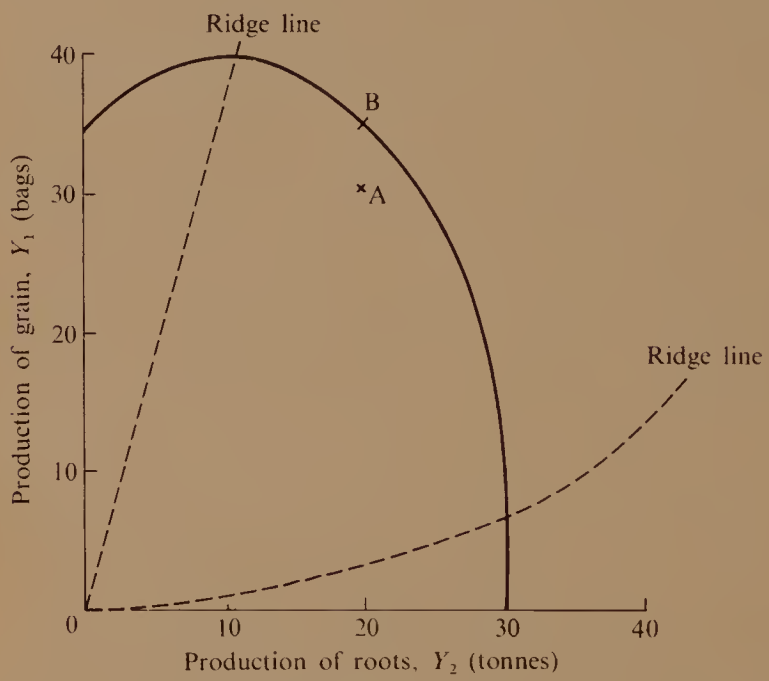


FIG. 1.7. The production-possibility curve. This is the boundary of all feasible combinations of grain and root production for the given resource limitations and production functions. Any point inside the curve, such as A, is feasible but technically inefficient in that more of both products may be obtained with the given resources. The slope is negative between the ridge lines but the curve is concave to the origin.

If less labour is available, or if labour is used inefficiently, the combination of roots and grain produced will lie inside the production-possibility curve. Conversely if more than 7 units of labour are available the new production-possibility curve will lie *outside* the curve for 7 units of labour. Certain characteristics of the production-possibility curve should be noted as they apply to most such curves for any pair of products.

First, as with isoquants, the slope of the curve is a measure of the rate at which one commodity can replace another, only in this case we are considering products rather than factors. As resources are transferred

from grain production to root-crop production, in effect grain is 'transformed' into roots. The rate at which this occurs is called 'the rate of product transformation'.

Secondly, it should be noted that over much of the curve the slope is negative: more roots can be produced only at the expense of less grain—the enterprises are 'competitive'. The rate of product transformation is therefore defined as the negative of the slope of the production-possibility curve. Unlike isoquants however, the production-possibility curve is concave to the origin, meaning that it bulges outwards. The rate of transformation from one product into another *increases* as more resources are transferred. As labour is moved into root-crop production the opportunity cost, in terms of grain forgone, increases as we have seen. At the same time marginal returns in root-crop production diminish.

We have already noted that the first unit of labour used in grain production does not have any opportunity cost because the marginal product of the seventh unit of labour used on root crops is zero. This means that the rate of product transformation is infinite; one product can be expanded without causing a reduction in the other product so the enterprises are said to be *supplementary*. Clearly supplementary enterprises do not compete for resources.

Another interesting section of the curve occurs between zero and 10 tonnes of root production. Over this section as root-crop output is increased so too is grain output. When the production of one enterprise increases as the other is increased, they are called *complementary enterprises* for as long as this relationship holds. In this example the enterprises are complementary because the marginal returns in grain production are negative. As less labour is used in grain production the total product actually increases. As we have seen, it would be irrational to use more than 5 units of labour in grain production, which now means that it would be irrational to use less than 2 units of labour on root crops or to produce less than 12 tonnes of roots. Again 'ridge lines' may be drawn (the broken lines in Fig. 1.7), separating the areas with negative slope (positive rate of product transformation) from the irrational areas.

Summary

The assumptions of this model of smooth, continuous production relationships, all of which are based on the law of diminishing returns, are as follows:

1. Factor–product response curves with positive but diminishing marginal products (i.e. concave response curves).

2. Factor–factor relationships with diminishing rates of technical substitution (i.e. convex isoquants).
3. Product–product relationships with increasing rates of product transformation (i.e. concave production-possibility boundary).

Mathematical appendix

One variable factor; one product

Let X denote the input of variable resource per unit of time and Y the total product per unit of time. Now, since for every value of X there is a

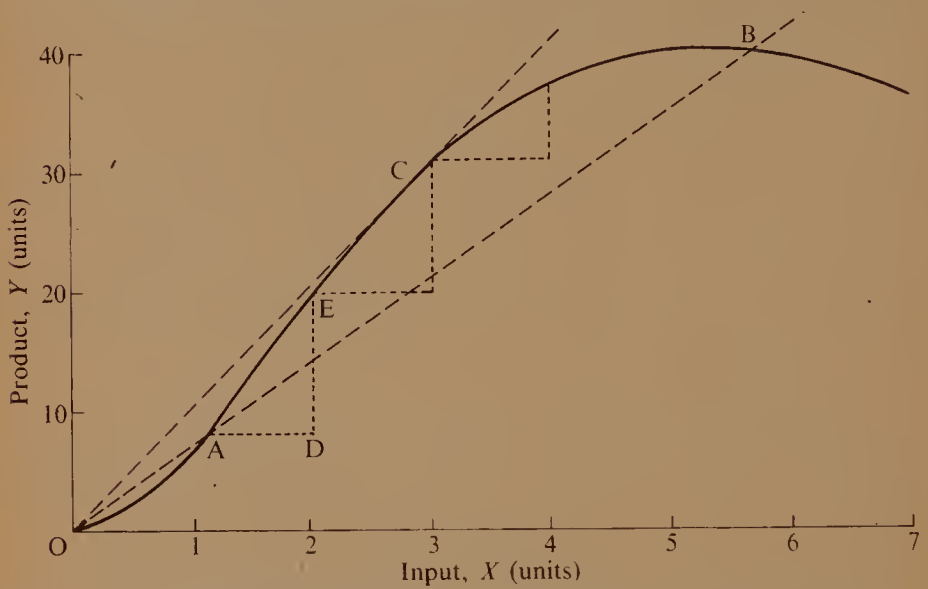


FIG. 1.8. The relationship between total, marginal, and average products. Marginal product is the slope of the total product curve, measured over an interval (e.g. between A and E the slope is ED/AD) or at a point (e.g. at point C the slope of the curve is equal to the slope of the tangent OC). From about 1·5 units of input upwards, the response curve is concave from below, i.e. slope (marginal product) is diminishing. At just over 5 units of input, slope (marginal product) is zero so total product is at its maximum. Beyond this point slope (marginal product) is negative.
Average product equals Y/X, i.e. the slope of the ray from the origin to the point in question. Average product is the same at A and B since they lie on the same ray. Average product is at a maximum at point C where it equals marginal product (i.e. the slopes of the ray from the origin and of the total product curve are equal).

corresponding value of Y we may assume that the two are related in some systematic manner. In fact we assume that Y depends upon X. In mathematical symbols we may write $Y = f(X)$ which means 'Y is a function of X' and does *not* represent some number X multiplied by another number f. This is the production function.

Such a production function is represented by the data given in Table 1.2, where the figures in column I are values of X and those in column II the corresponding values of Y. It may be possible to express the function by means of a formula such as:

$$Y = 3X^{0.4} \quad \text{or} \quad Y = 6 + 10X - 2X^2,$$

from which values of Y can be calculated for specific values of X. This is subject, of course, to the condition that X and Y cannot take negative values.

From the production function we can derive three quantities of interest:

(1) average product (AP);
(2) marginal product (MP); and
(3) elasticity of response (E).

The relationships between total product, (AP), and (MP) are illustrated in Fig. 1.8, which is simply an enlargement of the response curve from Fig. 1.1.

1. *Average product.* This is defined as

$$(AP) = \frac{Y}{X}$$

Thus (AP) at point A on the graph is $7/1 = 7$ bags of grain. This is the slope of the ray OA which connects point A with the origin. (AP) at point B is exactly the same since it lies on the same ray through the origin ($39{\cdot}9/5{\cdot}7 = 7$ bags of grain).

2. *Marginal product.* If we use the symbol d to mean a small change, so that dY is a change in Y and dX is a change in X, then marginal product is defined as

$$(MP) = \frac{\mathrm{d}Y}{\mathrm{d}X}.$$

This is the average slope of the total product curve itself over the interval X to $X + \mathrm{d}X$. For instance (MP) for the second unit of input is the slope of the line AE which equals

$$\frac{\mathrm{ED}}{\mathrm{AD}} = \frac{13}{1} = 13 \text{ bags of grain.}$$

At a given *point*, however, (MP) is the actual slope of the total product curve.

3. *Elasticity of response.* Elasticity of response to variation in X is measured as relative change in Y divided by the relative change in X thus:

$$E = \frac{\mathrm{d}Y}{Y} \Big/ \frac{\mathrm{d}X}{X}$$

which is equivalent to

$$E = \frac{\mathrm{d}Y}{\mathrm{d}X} \Big/ \frac{Y}{X} = \frac{(MP)}{(AP)}.$$

Since (MP) and (AP) are measured in the same units E is a pure number.

If (AP) has a maximum it occurs where the ray through the origin just touches (i.e. is a tangent to) the total product curve. In Fig. 1.8, this occurs at point C, the ray OC having the steepest slope of any average product line. At this point the slope of the ray (Y/X) is equal to the slope of the total product curve $(\mathrm{d}Y/\mathrm{d}X)$. In other words $(AP) = (MP)$ where (AP) is at a maximum.[8]

[8] Readers with a knowledge of differential calculus will note that for a given value of X, $(MP) = \delta Y/\delta X$, which at the limit as δX approaches zero becomes $\mathrm{d}Y/\mathrm{d}X$, the first derivative of Y with respect to X.

If the mathematical form of the function is known and if it is differentiable then (MP) can be derived directly.

Using the differential calculus

$$\frac{\mathrm{d}(AP)}{\mathrm{d}X} = \left(X\frac{\mathrm{d}Y}{\mathrm{d}X} - Y\right)\frac{1}{X^2} = \left(\frac{\mathrm{d}Y}{\mathrm{d}X} - \frac{Y}{X}\right)\frac{1}{X}.$$

We assume

$$X, \frac{\mathrm{d}Y}{\mathrm{d}X}, \text{ and } \frac{Y}{X} > 0.$$

Therefore if

$$\frac{\mathrm{d}Y}{\mathrm{d}X} > \frac{Y}{X} \text{ then } \frac{\mathrm{d}(AP)}{\mathrm{d}X} > 0$$

we have increasing average returns, and if

$$\frac{\mathrm{d}Y}{\mathrm{d}X} < \frac{Y}{X} \text{ then } \frac{\mathrm{d}(AP)}{\mathrm{d}X} < 0$$

we have diminishing average returns, so that if

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{Y}{X}, \quad \frac{\mathrm{d}(AP)}{\mathrm{d}X} = 0$$

where the maximum (or minimum) average product occurs.

At low levels of input, where (AP) and (MP) are increasing, the slope of the total product curve is steeper than the slope of ray of (AP)

$$\frac{dY}{dX} > \frac{Y}{X}, \qquad (MP) > (AP).$$

In stage II, at levels of input above the point of maximum (AP), both (AP) and (MP) are diminishing:[9]

$$\frac{dY}{dX} < \frac{Y}{X}; \qquad (MP) < (AP).$$

Maximum total product occurs where the slope $dY/dX = (MP)$ is zero. Before this point (MP) is positive so total product is increasing. Beyond this point (MP) is negative so total product is decreasing.

At maximum average product $(MP) = (AP)$ so $E = \dfrac{(MP)}{(AP)} = 1$

At maximum total product $(MP) = 0$ so $E = \dfrac{(MP)}{(AP)} = 0$

Thus over stage II $1 > E > 0$.

Two variable factors; one product

If we have two variable inputs X_1 and X_2 the production function may now be written: $Y = f(X_1, X_2)$. Again there may be a formula which will describe the function, such as

$$Y = 3X_1^{0.4}X_2^{0.6} \qquad \text{or}$$

$$Y = 6 + 10X_1 + 4X_2 - 2X_1^2 - 1X_2^2 + 3X_1X_2,$$

all Xs and Ys being non-negative. However, strictly speaking all inputs should be included in the function so the general form should be

$$Y = f(X_1, X_2, X_3, \ldots, X_n)$$

or, if we wish to show that all but the first two inputs are fixed,

$$Y = f(X_1, X_2/X_3, \ldots, X_n).$$

We now have two average products:

$$(AP)_1 = \frac{Y}{X_1} \qquad \text{and} \qquad (AP)_2 = \frac{Y}{X_2};$$

[9] In Stage II, $d^2Y/dX^2 < 0$ (diminishing marginal returns), though up to the end of this stage $dY/dX > 0$.

two marginal products:

$$(MP)_1 = \frac{dY}{dX_1} \qquad \text{and} \qquad (MP)_2 = \frac{dY}{dX_2};$$

and two elasticities of response

$$E_1 = \frac{(MP)_1}{(AP)_1} \qquad \text{and} \qquad E_2 = \frac{(MP)_2}{(AP)_2}.$$

The *rate* of *technical substitution* of resource 1 for resource 2 is defined as

$$(RTS)_{12} = -\frac{dX_2}{dX_1}.$$

If the input X_1 is increased by a small amount dX_1, other inputs being unchanged, we would expect total product Y to increase by

$$dX_1 \,.\, (MP)_1, \qquad \text{so} \qquad dY = dX_1 \,.\, (MP)_1.$$

Similarly, for input X_2, we have $dY = dX_2 \,.\, (MP)_2$ and therefore the total increase in Y is

$$dY = dX_1 \,.\, (MP)_1 + dX_2 \,.\, (MP)_2.$$

However we assume that, for movements along an isoquant, $dY = 0$, the increase in X_1 being balanced by a decrease in X_2, so

$$dX_1 \,.\, (MP)_1 = -dX_2 \,.\, (MP)_2.$$

Dividing both sides by $dX_1 \,.\, (MP)_2$ gives

$$\frac{(MP)_1}{(MP)_2} = -\frac{dX_2}{dX_1} = (RTS)_{12}.$$

That is, the rate of technical substitution equals the ratio of the marginal products.

Note that as the input X_1 is increased, $(MP)_1$ will decrease (according to the law of diminishing returns). But if Y is constant an increase in X_1 is associated with a decrease in X_2 so $(MP)_2$ will increase (diminishing returns in reverse). Hence if in the fraction

$$\frac{(MP)_1}{(MP)_2} = (RTS)_{12}$$

the numerator is decreasing and the denominator is increasing the value of $(RTS)_{12}$ must be diminishing. The law of diminishing returns implies diminishing rates of technical substitution.[10]

Returns to scale

To simplify the discussion we will assume only two inputs, though the argument is readily extended to include more.

Let us consider a particular point A on the production surface, where $Y^A = f(X_1{}^A, X_2{}^A)$, and assume that both inputs are increased in the same proportion $\mathrm{d}l/l$, so that

$$\frac{\mathrm{d}l}{l} = \frac{\mathrm{d}X_1}{X_1{}^A} = \frac{\mathrm{d}X_2}{X_2{}^A}$$

or, by rearrangement,

$$\mathrm{d}X_1 = \frac{\mathrm{d}l}{l} \cdot X_1{}^A \qquad \text{and} \qquad \mathrm{d}X_2 = \frac{\mathrm{d}l}{l} \cdot X_2{}^A. \tag{1.1}$$

(Note that this implies that factor proportions X_1/X_2 stay constant.) Now, provided that $\mathrm{d}X_1$ and $\mathrm{d}X_2$ are small enough the resultant change in output $\mathrm{d}Y$ is given by

$$\mathrm{d}Y = \mathrm{d}X_1 \,.\, (MP)_1 + \mathrm{d}X_2 \,.\, (MP)_2$$

or by substituting from eqns (1.1):

$$\mathrm{d}Y = \frac{\mathrm{d}l}{l} \,.\, X_1{}^A \,.\, (MP)_1 + \frac{\mathrm{d}l}{l} \,.\, X_2{}^A \,.\, (MP)_2. \tag{1.2}$$

We then divide through by Y^A to arrive at the proportionate change in output:

$$\frac{\mathrm{d}Y}{Y^A} = \frac{\mathrm{d}l}{l} \cdot \frac{X_1{}^A}{Y^A} \,.\, (MP)_1 + \frac{\mathrm{d}l}{l} \cdot \frac{X_2{}^A}{Y^A} \cdot (MP)_2. \tag{1.3}$$

[10] Differentiating $(RTS)_{12}$ with respect to X_1 gives

$$\frac{\mathrm{d}(RTS)_{12}}{\mathrm{d}X_1} = \frac{\mathrm{d}\{(MP)_1/(MP)_2\}}{\mathrm{d}X_1} = \left\{(MP)_2 \frac{\mathrm{d}(MP)_1}{\mathrm{d}X_1} - (MP)_1 \frac{\mathrm{d}(MP)_2}{\mathrm{d}X_1}\right\} \frac{1}{(MP)_2{}^2},$$

which then, substituting the total derivatives of $(MP)_1$ and $(MP)_2$ with respect to X_1,

$$= \{(MP)_2{}^2(\partial^2 Y/\partial X_1{}^2) - 2(MP)_1(MP)_2(\partial^2 Y/\partial X_1 \partial X_2) + (MP)_1{}^2(\partial^2 Y/\partial X_2{}^2)\}/(MP)_2{}^3.$$

If $(MP)_1 > 0$ and $(MP)_2 > 0$ while $\partial^2 Y/\partial X_1{}^2 < 0$, $\partial^2 Y/\partial X_2{}^2 < 0$, and $\partial^2 Y/\partial X_1 \partial X_2 > 0$ (positive interaction) it follows that $\mathrm{d}(RTS)_{12}/\mathrm{d}X_1 < 0$.

But $Y^A/X_1{}^A = (AP)_1$ at point A by definition, and $(MP)_1/(AP)_1 = E_1$, also by definition. Hence

$$\frac{X_1{}^A}{Y^A} \cdot (MP)_1 = \frac{(MP)_1}{(AP)_1} = E_1;$$

likewise

$$\frac{X_2{}^A}{Y^A} \cdot (MP)_2 = E_2.$$

Substitution in eqn (1.3) now gives

$$\frac{\mathrm{d}Y}{Y^A} = \frac{\mathrm{d}l}{l} E_1 + \frac{\mathrm{d}l}{l} E_2 = \frac{\mathrm{d}l}{l} (E_1 + E_2),$$

or, if we now define the 'scale coefficient' e as

$$e = E_1 + E_2,$$

then

$$\frac{\mathrm{d}Y}{Y^A} = \frac{\mathrm{d}l}{l} \cdot e. \tag{1.4}$$

Thus the proportionate change in output $\mathrm{d}Y/Y^A$ is greater than, equal to, or less than the proportionate change in inputs $\mathrm{d}l/l$ according to whether e is greater than, equal to, or less than 1, at point A.

If e is constant for all levels of output and for any given factor proportions, the production function is said to be homogeneous. Thus if there are constant returns to scale everywhere, e is always equal to 1 so the function is said to be 'homogeneous of degree 1' or 'linearly homogeneous'. Where this is the case, since

$$\frac{\mathrm{d}Y}{Y} = \frac{\mathrm{d}l}{l} = \frac{\mathrm{d}X_1}{X_1} = \frac{\mathrm{d}X_2}{X_2}$$

everywhere,

$$\frac{Y}{X_1} = (AP)_1 \qquad \text{and} \qquad \frac{Y}{X_2} = (AP)_2$$

are constant for any increase in scale. For this to be so,

$$\frac{\mathrm{d}Y}{\mathrm{d}X_1} = (MP)_1 \qquad \text{and} \qquad \frac{\mathrm{d}Y}{\mathrm{d}X_2} = (MP)_2$$

must also be constant.[11]

[11] Strictly speaking a production function $Y = f(X_1, X_2)$ is defined as being homogeneous of degree e if

$$f(lX_1, lX_2) = l^e \cdot f(X_1, X_2).$$

Product exhaustion under constant returns to scale

For a homogeneous function eqn (1.4) is generally applicable, so it may be written

$$\frac{\mathrm{d}Y}{Y} = \frac{\mathrm{d}l}{l} \cdot e,$$

and therefore

$$\mathrm{d}Y = \frac{\mathrm{d}l}{l} \cdot e \cdot Y.$$

Substitution in eqn (1.2), applied generally, gives

$$\frac{\mathrm{d}l}{l} \cdot e \cdot Y = \frac{\mathrm{d}l}{l} \cdot X_1 \cdot (MP)_1 + \frac{\mathrm{d}l}{l} \cdot X_2 \cdot (MP)_2.$$

Therefore for homogeneous functions

$$e \cdot Y = X_1 \cdot (MP)_1 + X_2 \cdot (MP)_2$$

(this is known as Euler's theorem). Thus if $e = 1$ we have

$$Y = X_1 \cdot (MP)_1 + X_2 \cdot (MP)_2,$$

i.e. total product = quantity of factor 1 times its marginal product plus quantity of factor 2 times its marginal product.

Now if we let $Y^{\mathrm{B}} = f(lX_1, lX_2)$ and $Y^{\mathrm{A}} = f(X_1, X_2)$ in this equation and differentiate partially with respect to X_1 and X_2, we obtain:

$$l \cdot \frac{\partial Y^{\mathrm{B}}}{\partial X_1} = l^e \cdot \frac{\partial Y^{\mathrm{A}}}{\partial X_1} \quad \text{and} \quad l \cdot \frac{\partial Y^{\mathrm{B}}}{\partial X_2} = l^e \cdot \frac{\partial Y^{\mathrm{A}}}{\partial X_2},$$

i.e.

$$l \cdot (MP)_1{}^{\mathrm{B}} = l^e \cdot (MP)_1{}^{\mathrm{A}} \quad \text{and} \quad l \cdot (MP)_2{}^{\mathrm{B}} = l^e (MP)_2{}^{\mathrm{A}}.$$

(*i*) By taking the ratio of these two equations we have for all homogeneous functions

$$\frac{(MP)_1{}^{\mathrm{B}}}{(MP)_2{}^{\mathrm{B}}} = \frac{(MP)_1{}^{\mathrm{A}}}{(MP)_2{}^{\mathrm{A}}},$$

i.e. $(RTS)_{12}{}^{\mathrm{B}} = (RTS)_{12}{}^{\mathrm{A}}$; the rate of technical substitution is constant when inputs are increased in strict proportion.

(*ii*) With constant returns to scale when $e = 1$,

$$(MP)_1{}^{\mathrm{B}} = (MP)_1{}^{\mathrm{A}} \quad \text{and} \quad (MP)_2{}^{\mathrm{B}} = (MP)_2{}^{\mathrm{A}}:$$

marginal products remain constant when inputs are increased in strict proportion.

One variable factor; two products

We now assume one variable factor is allocated between two products. Other factors are specific to one product, and fixed. Thus we have two functions:

$$Y_1 = f_1(X_1/X_2, X_3, \ldots, X_n)$$

and

$$Y_2 = f_2(X_1/X_{n+1}, X_{n+2}, \ldots, X_m).$$

The variable factor now has two average products:

$$(AP, Y_1) = Y_1/X_1 \qquad \text{and} \qquad (AP, Y_2) = Y_2/X_1;$$

and two marginal products:

$$(MP, Y_1) = \mathrm{d}Y_1/\mathrm{d}X_1 \qquad \text{and} \qquad (MP, Y_2) = \mathrm{d}Y_2/\mathrm{d}X_1.$$

The rate of product transformation into product 2 from product 1 is defined as

$$(RPT)_{21} = -\frac{\mathrm{d}Y_1}{\mathrm{d}Y_2}.$$

Note that for competing products an increase in Y_2 must be associated with a decrease in Y_1 so $\mathrm{d}Y_1/\mathrm{d}Y_2$ is negative and $(RPT)_{21}$ is positive.

Now, if the use of the variable factor on product 2 is increased by a small amount $\mathrm{d}X$ then, for product 2, $\mathrm{d}X_1 = \mathrm{d}X$. However, since the total amount of the factor is fixed, there must be a corresponding *decrease* in the amount used on product 1. Hence for product 1, $\mathrm{d}X_1 = -\mathrm{d}X$.

The resultant changes in the outputs of the two products are

$$\mathrm{d}Y_1 = -\mathrm{d}X \,.\, (MP, Y_1)$$

and

$$\mathrm{d}Y_2 = \mathrm{d}X \,.\, (MP, Y_2).$$

Dividing the first equation by the second and cancelling $\mathrm{d}X$ gives

$$\frac{\mathrm{d}Y_1}{\mathrm{d}Y_2} = -\frac{(MP, Y_1)}{(MP, Y_2)}.$$

But $-\mathrm{d}Y_1/\mathrm{d}Y_2 = (RPT)_{21}$ by definition (see above), so

$$(RPT)_{21} = \frac{(MP, Y_1)}{(MP, Y_2)}.$$

i.e. rate of product transformation equals the ratio of the two marginal products.

As the use of the variable factor is increased for product 2, its marginal product will diminish (by the law of diminishing marginal returns). But at the same time its use on product 1 is decreased so its marginal product will increase (diminishing marginal returns in reverse). Hence in the fraction

$$\frac{(MP, Y_1)}{(MP, Y_2)} = (RPT)_{21},$$

the numerator is increasing and denominator diminishing, so the value of $(RPT)_{21}$ must be increasing, i.e. the law of diminishing marginal returns implies increasing rates of product transformation.[12]

[12] $$\frac{\mathrm{d}(RPT)_{21}}{\mathrm{d}Y_2} = \frac{\mathrm{d}}{\mathrm{d}X}\left\{\frac{(MP, Y_1)/(MP, Y_2)}{\mathrm{d}Y_2/\mathrm{d}X}\right\}$$

$$= \frac{\{(MP, Y_2)\mathrm{d}(MP, Y_1)/\mathrm{d}X - (MP, Y_1)\mathrm{d}(MP, Y_2)/\mathrm{d}X\}/(MP, Y_2)^2}{\mathrm{d}Y_2/\mathrm{d}X}$$

$$= \frac{\{(MP, Y_2)(-\mathrm{d}^2 Y_1/\mathrm{d}X_1^2) - (MP, Y_1)\mathrm{d}^2 Y_2/\mathrm{d}X_1^2\}/(MP, Y_2)^2}{\mathrm{d}Y_2/\mathrm{d}X_1}$$

since for Y_1, $\mathrm{d}X = -\mathrm{d}X_1$ and for Y_2, $\mathrm{d}X = \mathrm{d}X_1$. But

$$(MP, Y_1) > 0 \quad \text{and} \quad (MP, Y_2) = \frac{\mathrm{d}Y_2}{\mathrm{d}X_1} > 0$$

(positive marginal products), while

$$\frac{\mathrm{d}^2 Y_1}{\mathrm{d}X_1^2} < 0 \quad \text{and} \quad \frac{\mathrm{d}^2 Y_2}{\mathrm{d}X_1^2} < 0$$

(diminishing marginal returns). Therefore

$$\frac{\mathrm{d}(RPT)_{21}}{\mathrm{d}Y_2} > 0$$

(increasing rate of product transformation).

Suggestions for further reading

DILLON, J. L. (1968). *The analysis of response in crop and livestock production.* Pergamon Press, Oxford.

DOLL, J. P., RHODES, V. J., and WEST J. G. (1968). *Economics of agricultural production, markets and policy.* Irwin, Homewood, Illinois.

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2. Linear production relationships

Fixed-factor proportions

This chapter deals with an alternative theory of production based on linear, i.e. straight-line, relationships. The essential idea is that of a 'process' or 'activity' which is a specific method of producing a particular product. For example, wheat production may be viewed as an activity, but if there are several different alternative ways of producing wheat, for example winter wheat or spring wheat, wheat production using owned machinery or wheat production using hired machinery, then each method represents a separate activity. If the wheat may be either sold at harvest or stored, then wheat storage may be viewed as a separate activity. The reason for this rather narrow definition of an activity is the assumption that factor inputs are always combined in specific fixed proportions for a given activity, regardless of the level of output. We assume that for a given method of wheat production every hectare requires the same amount of labour time, machinery time, fertilizer, seed, sprays, and so on. In other words, an individual activity can only be increased to scale.

Furthermore we assume constant returns to scale, so that every hectare of wheat yields the same amount of grain no matter how many hectares are grown by a particular process. For example, if 1 hectare of grain production yields 12 bags and requires 0·5 unit of labour, then we assume 1000 hectares would yield 12 000 bags of grain and require 500 units of labour. Similarly 1/10 hectare of grain is assumed to yield 1·2 bags of grain and to require 1/20 of a unit of labour.

Thus, if we assume that very large amounts of all resources, other than labour are available, so that grain production can be expanded to scale, the relationship between increasing labour input and product output might be as shown in Fig. 2.1(a). This is drawn on the assumption of a yield per hectare of 12 bags, a labour requirement of 0·5 unit per hectare, and a total of 5 units of labour available. The maximum possible output is now limited by the total amount of labour available, which is therefore called a 'constraint'. The constraint becomes 'effective' when all 5 units of labour are in use. This means that for each of the 5 units of labour the average product is the same, namely $12/0{\cdot}5 = 24$ bags of grain per unit of labour. Furthermore, provided there are no other effective constraints the marginal product: i.e. the addition to the total product for each

additional unit of labour is also 24 bags of grain. Marginal product equals average product and is constant as shown by the horizontal line Fig. 2.1(b).[1]

The situation is different, however, if other resources are limited and the amounts available become effective constraints before all the labour

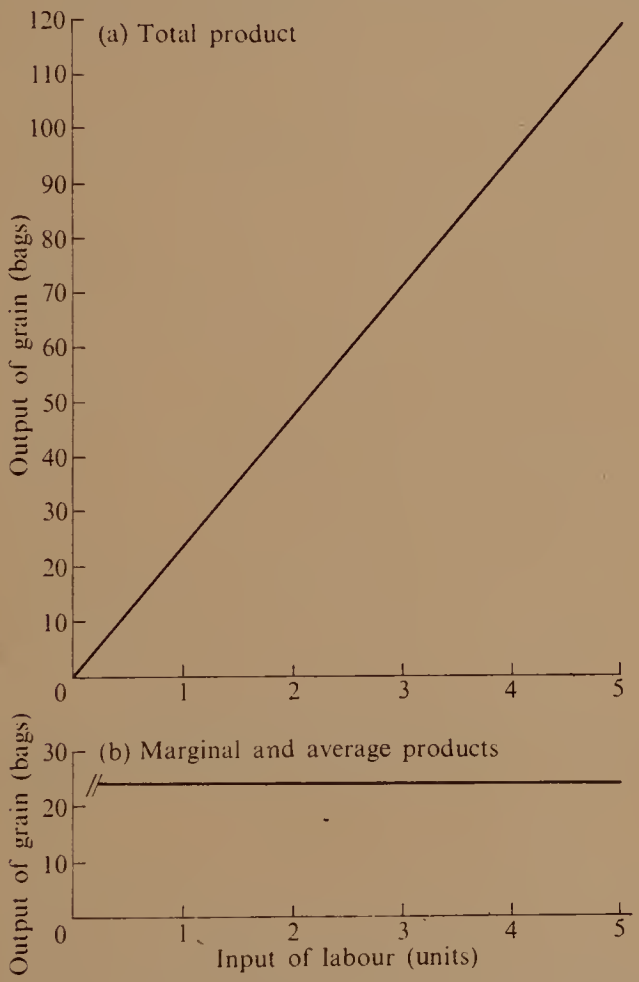


FIG. 2.1. Linear response. Provided that there are no other effective constraints the relationship between labour input and grain output is given by a straight line. The slope which measures both marginal and average product, is therefore constant.

[1] Note that labour is not the only input being varied. Strictly speaking, the marginal product per unit of a variable factor is defined when the quantities of all other factors *in use* are held constant. Under the present assumptions this would mean that all marginal products would be zero. Thus we now change the definition of marginal product slightly to mean the extra product obtained per unit of a variable factor when the quantities *available* of all other factors are held constant.

is in use. For instance, if only 3 hectares of land were available for grain production, using 0·5 unit of labour per hectare, then only $3 \times 0{\cdot}5 = 1{\cdot}5$ units of labour could be employed productively in this activity. If there were no alternative activities, and 5 units of labour were available, then 3·5 units would be unused, when they are said to be 'in disposal'. At a yield per hectare of 12 bags, total product would be limited to $3 \times 12 = 36$ bags. Thus the total product-response relationship will appear as in Fig. 2.2. The first 1·5 units of labour yield an average and a marginal

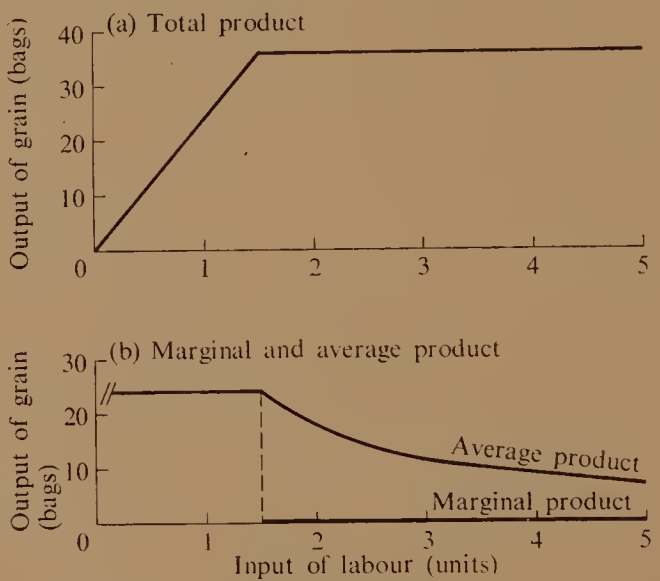


FIG. 2.2. (a) The effect of a constraint. Once the constraint becomes effective no further expansion of the activity is possible. (b) The marginal product of additional labour then falls to zero and the average product gradually diminishes.

product per unit of 24 bags of grain, but then there is a sudden change. Beyond 1·5 units of input the marginal product per additional unit of labour is zero because there is no extra land on which to employ it. Average product per unit of labour therefore falls gradually: at 1·5 units of labour input it is $36/1{\cdot}5 = 24$, at 2 units it is $36/2 = 18$, and at 5 units it is $36/5 = 7{\cdot}2$ bags of grain. Thus in practice marginal and average products of a variable input may diminish if some other factor limitations are effective constraints.

The isoquant for a single activity is as shown in Fig. 2.3, where the two inputs are land and labour. Because of the assumption that factor inputs must be combined in fixed proportions, it is not possible to substitute one input for the other. For a given level of output, the amount of land

required cannot be reduced by increasing labour inputs. Similarly, labour cannot be saved by increasing land-use per unit of output without a change of technique, i.e. without changing to a different activity. Hence for a single activity the isoquants are right-angled. The activity is represented by the ray through the origin, its slope representing the fixed proportion in which the inputs are combined. Because there are constant returns to scale, the isoquants are equally spaced. The relationship

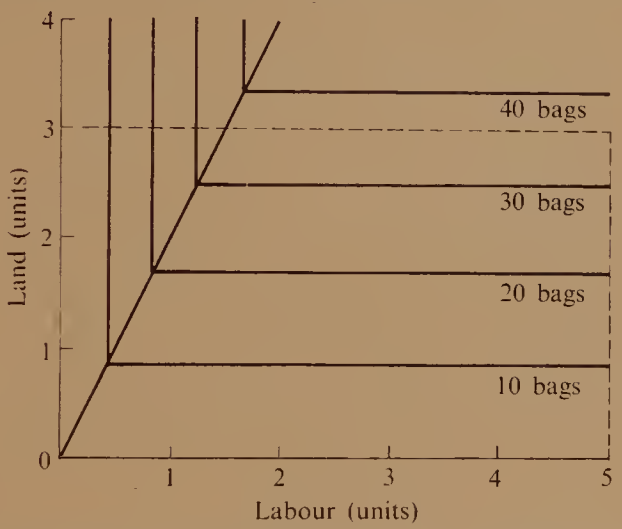


FIG. 2.3. Isoquants for a single activity. For a single, fixed-factor-proportions activity no substitution of land for labour, or vice versa, is possible. For the constraint levels shown by the broken lines, we see that land is the effective constraint.

between both inputs and output is illustrated in three dimensions in Fig. 2.4.

Factors which cannot be substituted for each other and which therefore have right-angled isoquants are said to be complementary: one is a complement to the other. The assumption of complementary relationships between factors of production may be more realistic than the assumption of smooth substitutability discussed in the previous chapter. In practice it may be impossible to substitute land for labour, to substitute tractors for tractor-drivers, or to substitute milking machines for cattle feed in a given productive activity.

More than one activity

However, it is clear that some factor substitution must be possible. After all, one of the main reasons for using machinery is to save labour.

This possibility can be incorporated into the theory, if we assume that the farmer has a choice of several alternative activities and that these activities use factors in different proportions. Factor substitution is then achieved by activity substitution. For instance, let us assume that the activity we have been discussing, grain production, is a relatively unintensive system of alternating grain and fallow. Over a period of years it requires on average relatively little labour per hectare and yields relatively low returns. We shall call this activity 1. An alternative method of growing spring-sown grain every year, activity 2, requires more labour per

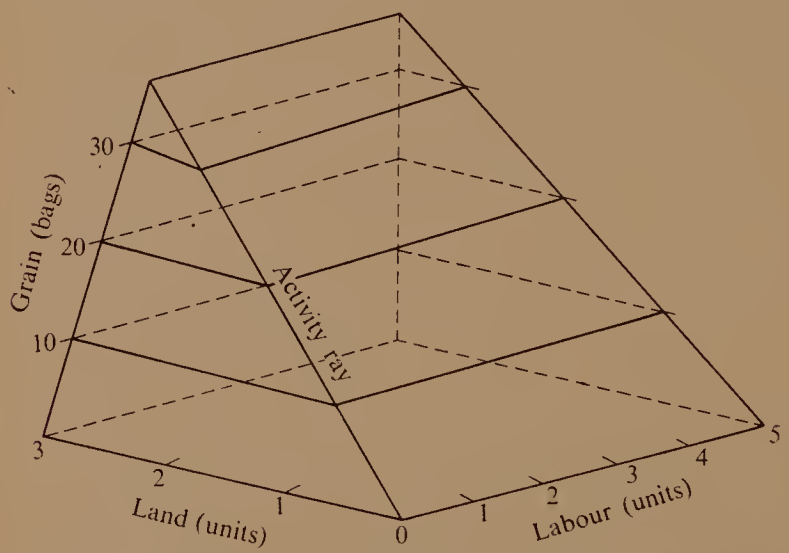


FIG. 2.4. A single activity in three-dimensions. This is really a three-dimensional version of Fig. 2.3. It may be compared with the production surface for a curved production function shown in Fig. 1.5 (p. 16).

hectare per year, say 0·75 units, and yields a higher return, say 16 bags. A third activity, activity 3, winter-sown grain grown every year, requires 2 units of labour per hectare and yields 25 bags. This information is set out in Table 2.1. Clearly activity 2 requires less land and more labour, per bag of grain produced, than does activity 1. Hence, if activity 2 is substituted for activity 1, in effect labour is substituted for land. A change to activity 3 takes the substitution a stage further.

Now, rather than substituting one activity for another entirely, it may be possible to operate a combination of two or more activities. Naturally, if a combination of activities would cause organizational chaos they would have to be mutually exclusive. This situation of mutual exclusiveness will be discussed further in Chapter 7 on capital. However, for present purposes we assume that such combinations are possible and

furthermore that there are no interactions, so that the inputs and outputs resulting from simultaneous use of two or more activities at any levels can be found by adding the inputs and outputs of the individual activities. Technically, such combinations are said to be 'linear combinations'.

On the strength of these assumptions we may now draw isoquants

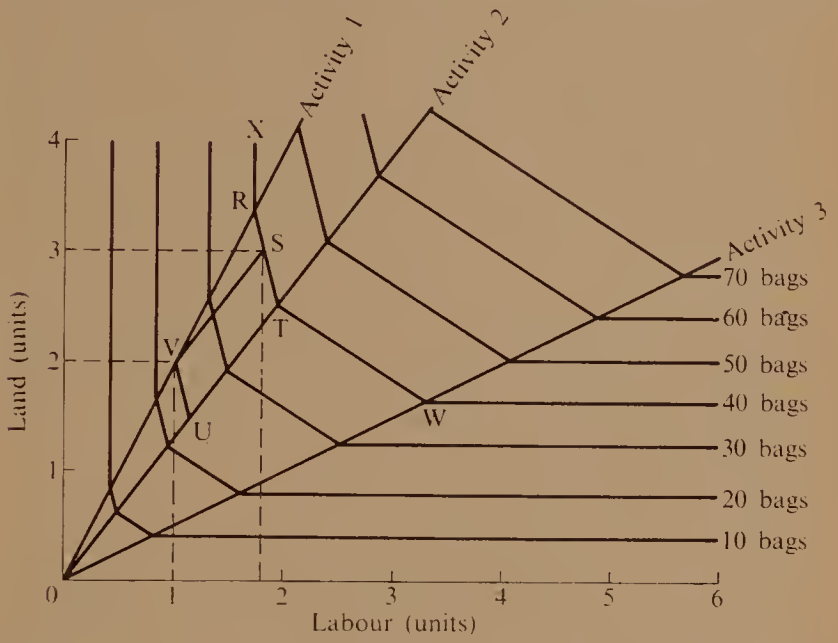


FIG. 2.5. Linear isoquants. With 2 or more activities, using factors in differing proportions, factor substitution is possible. By changing from activity 1 to activity 2 and then to activity 3, labour is substituted for land. Combinations of activities are also assumed to be possible. For instance 40 bags of grain may be produced by activity 1 alone at point R or by a combination of activities 1 and 2 such as that represented by point S. The combination of land and labour represented by point X is feasible but it is technically inefficient since RX units of land are in disposal or wasted.

TABLE 2.1

Inputs and outputs for three activities

	Activity 1	Activity 2	Activity 3
Grain output (bags)	12	16	25
Land input (hectares)	1	1	1
Labour input (units)	0·5	0·75	2

made up of linear segments as shown in Fig. 2.5. To see how they are constructed let us concentrate on the 40-bag isoquant for activities 1 and 2. If activity 1 is used alone then $40/12 = 3{\cdot}33$ hectares of land and

$3{\cdot}33 \times 0{\cdot}5 = 1{\cdot}67$ units of labour are needed (point R in Fig. 2.5) whereas if activity 2 is used alone $40/16 = 2{\cdot}5$ hectares of land and $2{\cdot}5 \times 0{\cdot}75 = 1{\cdot}875$ units of labour are needed (point T in Fig. 2.5).[2] However, we have assumed that 40 bags of grain can also be produced by the simultaneous use of both techniques in suitable proportions. Hence 40 bags of grain can be produced with any combination of land and labour represented on the straight line RT, by using both activities 1 and 2. For instance point S, which lies on the line RT, represents the production of 40 bags of grain from 3 hectares of land with 1·75 units of labour. To see how this may be achieved we draw the line SV parallel to OT (the ray for activity 2). This shows that OV = 24 bags of grain must be produced by activity 1, the line UV being a part of the 24-bag isoquant, and VS = UT = 40 − 24 = 16 bags must be produced by activity 2. We may check both from the data given in Table 2.1 and from Fig. 2.5 that this combination of activities requires 3 hectares of land (2 for activity 1 and 1 for activity 2), and 1·75 units of labour (1 for activity 1 and 0·75 for activity 2). Similarly it can be shown that any other point on the line RT represents the simultaneous use of both activities in suitable proportions.

Along the line RT, land-use falls from 3·3 hectares at point R to 2·5 hectares at point T, while labour-use increases from 1·67 units to 1·875 units. Hence the rate of technical substitution of labour for land, which is the decrease in land-use divided by the increase in labour-use, is calculated to be $(3{\cdot}3 - 2{\cdot}5)/(1{\cdot}875 - 1{\cdot}67) = 4{\cdot}0$. Turning now to activity 3 we find that 40 bags of grain require $40/25 = 1{\cdot}6$ hectares of land and $1{\cdot}6 \times 2 = 3{\cdot}2$ units of labour, as shown at point W in Fig. 2.5. Hence the straight line TW represents another segment of the 40-bag isoquant. The rate of technical substitution between point T and point W is $(2{\cdot}5 - 1{\cdot}6)/(3{\cdot}2 - 1{\cdot}875) = 0{\cdot}68$. Points on the 40- and 50-bag isoquants, together with rates of technical substitution are given in Table 2.2. Isoquants for any specific level of output can be derived in a similar manner.

There are various characteristics of these isoquants derived from linear relationships which should be noted. First, it would be irrational to use more than 3·33 hectares of land to produce 40 bags of grain output, or 4·17 hectares of land to produce 50 bags of grain output, because there is no alternative activity which uses more land and less labour per unit of

[2] Note that there is no reason to expect that point T will be the same distance from the origin as point R, although when output is measured vertically to the page they must both be the same height. However, on the two-dimensional isoquant diagram, the scale along each ray reflects the productivity of the factors in that activity and will generally differ from one activity to another.

output than activity 1. It is impossible to substitute land for labour beyond the ratio of 2 hectares per unit of labour in which the two factors are combined in activity 1. One could argue that it would take an infinitely large amount of land to substitute for a unit of labour, so the rate of technical substitution is infinite. Thus in Fig. 2.5 the area above and to the left of the ray for activity 1 is an irrational area so the ray for activity 1 is also a ridge line. Similarly the area below and to the right of the ray for activity 3 is also irrational, because labour cannot be substituted for land; the rate of technical substitution is zero. Hence the ray for activity 3 is the lower ridge line.

Secondly, between any pair of rays, the rate of technical substitution is constant for all levels of output. Thus in Table 2.2, the rate of technical

TABLE 2.2

Activity substitution; isoquants for 40 bags and 50 bags of grain

	Land	Decrease in land	Labour	Increase in labour	Rate of technical substitution (hectares of land per unit of labour)
	(hectares)		(units)		
(a) 40 bags of grain					
Activity 1	3·33		1·67		
		0·83		0·205	4·00
Activity 2	2·5		1·875		
		0·9		1·32	0·68
Activity 3	1·6		3·2		
(b) 50 bags of grain					
Activity 1	4·167		2·08		
		1·042		0·26	4·00
Activity 2	3·125		2·34		
				1·66	0·68
Activity 3	2	1·125	4		

substitution between activity 1 and activity 2 is given as 4·0 hectares of land per unit of labour for both the 40-bag and the 50-bag isoquant. In fact, because there are constant returns to scale in each activity, the isoquants are equally spaced along the rays and hence are parallel to each other.

Thirdly, in contrast with the smooth isoquants of Fig. 1.4 there are points where the rate of technical substitution is not defined. These points lie on the activity rays where kinks or angles occur in the isoquants. Nevertheless, subject to the above qualification, these isoquants are similar to those of the previous chapter in that the slope is negative (the

rate of technical substitution is therefore positive) and they are convex to the origin (the rate of technical substitution diminishes).[3] The similarity is even closer when there are large numbers of alternative activities, as shown in Fig. 2.6.

Naturally an activity which uses more of both factors per unit of output than some other activity would never be used. The inefficient activity is then said to be 'dominated' by the more efficient one. Likewise an activity which uses more of both factors, per unit of output, than some

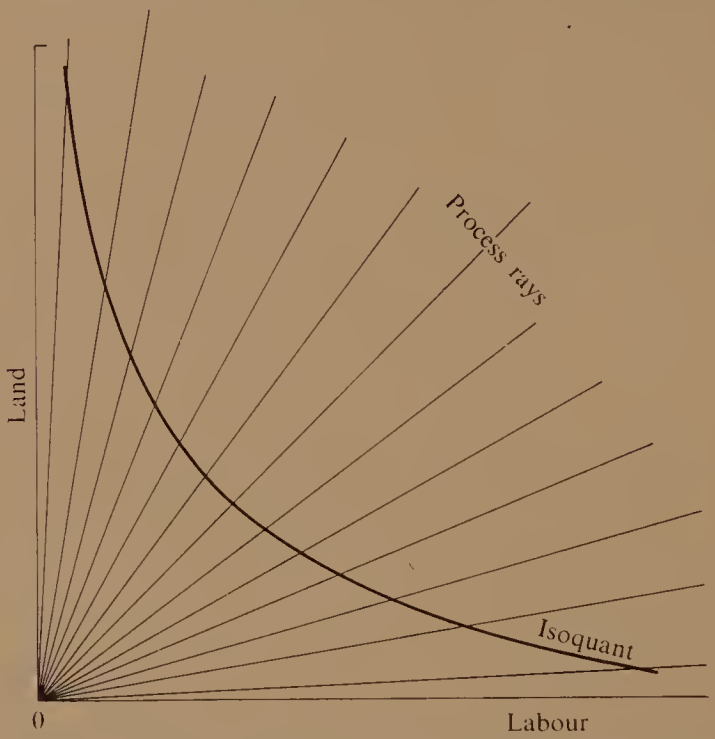


FIG. 2.6. Ease of substitution depends upon the number of activities. With many alternative activities all differing in their necessary factor proportions we get isoquants very similar to those obtained from smoothly curved production functions.

combination of other activities is technically inefficient and will never be used. For instance if there were a fourth activity which used inputs in the same proportions as activity 2, but which only yielded 13 bags of grain per hectare it would clearly be dominated by activity 2. However, even if activity 2 is excluded from the alternatives, activity 4 is still inefficient in that certain combinations of activity 1 with activity 3 require less of

[3] Strictly speaking, the rate of technical substitution is non-increasing, which allows for the possibility of it being constant as in Fig. 1.6.

both inputs per unit of output as shown in Fig. 2.7. Points R, T′, and W all represent 40 bags of grain output, but point T′ does not lie on the 40-bag isoquant. (Note that if it did the isoquant would not be convex to the origin; the rate of technical substitution would increase between point R and point W.) Some of the combinations of activities 1 and 3 represented by the straight line RW require less of both inputs than do combinations including activity 4 represented by points on RT′ or T′W. Hence activity 4 should be left out of our calculations.

It further follows that the only efficient combinations are of processes

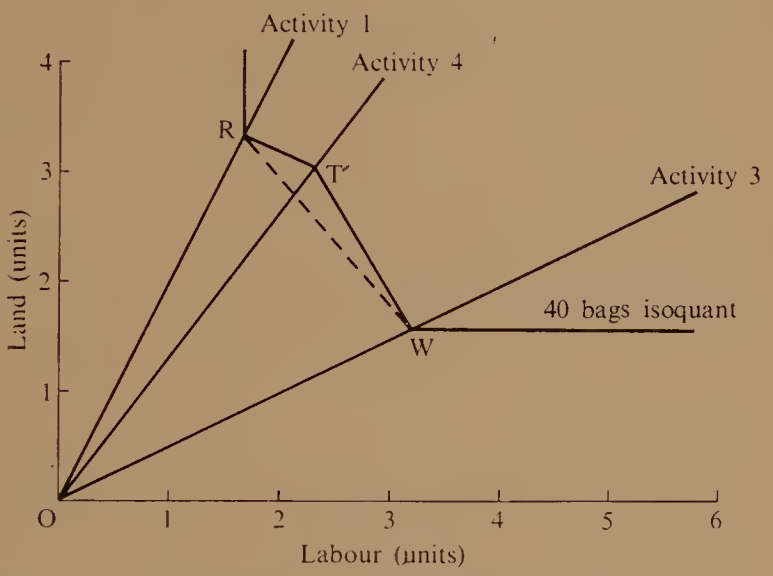


FIG. 2.7. An inefficient activity. Activity 4 requires more land and labour to produce 40 bags of grain (point T′) than does a combination of activities 1 and 3. Hence activity 4 is technically inefficient and may be ignored.

adjacent to one another. For instance it should be apparent from Fig. 2.5 that combinations of activities 1 and 3 must be less efficient than combinations of activities 1 and 2 or 2 and 3.

Constraints and the feasible region

So far we have discussed combinations of activities as though there were *no* effective resource limitations. If we now assume that only 3 hectares of land and only 5 units of labour are available, these constraints define the so-called feasible region. Combinations of activities that require more than 3 hectares of land or more than 5 units of labour are infeasible. Thus these constraints set a limit on the total amount of grain which can be produced. The maximum amount that can be produced

subject to the constraint limitations is the technical optimum and a combination of activities which produces less than the maximum output is technically inefficient.

Initially we may consider the individual activities to discover which of them will yield the maximum feasible output. The relevant estimates are given in Table 2.3. Output per hectare of land for each activity is taken directly from Table 2.1. Output per unit of labour is obtained simply by dividing output per hectare by the labour requirement per hectare. It is then a simple matter to calculate the total output of grain that would be

TABLE 2.3

Possible outputs for individual activities

	Activity 1	Activity 2	Activity 3
Grain output per hectare of land (bags)	12	16	25
Grain output per unit of labour (bags)	24	21·33	12·5
Grain output from 3 hectares of land (bags)	36	48	75
Grain output from 5 units of labour (bags)	120	106·67	62·5

obtained if all 3 hectares of land or all 5 units of labour were devoted to each individual activity.

Several relationships are illustrated by the figures in this table. Firstly, if there were only one constraint the maximum feasible product would be obtained from one activity alone. If the land limitation was the only constraint then clearly activity 3 would yield the maximum feasible product of 75 bags of grain. If the labour limitation was the only constraint then activity 1 would yield the maximum feasible product of 120 bags of grain.

In order to decide which activity will yield the maximum output when there are two (or more) constraints, we must first check which constraint is the effective one for each of the activities. As we have already seen, it is unlikely that more than one constraint will be effective for a single activity. The effective constraint sets a limit on the output which can be produced with that activity. The activities are now compared to find the over-all maximum output which can be produced from a single activity. Table 2.3 shows that land is the effective constraint for activity 1, since the lowest of the feasible outputs is 36 bags, produced when all 3 hectares of land are devoted to that activity. Similarly, the land constraint is effective in limiting output from activity 2 to 48 bags. However, for activity 3, it is the labour constraint which limits output to 62·5 bags.

The over-all maximum output is now the highest of these three feasible outputs, namely the 62·5 bags from activity 3 alone, using all 5 units of labour and 2·5 hectares of land, so that some land is left in disposal.

If the activities were mutually exclusive, then the 62·5 bags of grain produced from 2·5 hectares of activity 3 would represent the maximum possible output, or technical optimum. However, if activities can be combined it may be possible to improve on this. In particular, one suspects that if the land left in disposal by this single activity plan could be brought into productive use, output would be increased. However, it should not be assumed that in order to maximize output every factor of production should be used up to the limit so that none is left in disposal. Indeed, this may be impossible to achieve, if the number of constraints exceeds the number of alternative activities. However, in this case the number of alternative activities exceeds the number of constraints so it is less likely that resources will remain in disposal when output is maximized.

Clearly the output of activity 3 cannot be increased beyond 62·5 bags of grain unless the quantity of labour available can be increased above 5 units. Nevertheless it may be possible to increase output by introducing activity 1 or activity 2 even though the labour constraint is fixed. Let us consider the effect of adding 1 hectare of activity 1, which requires 0·5 units of labour.[4] In order to release this labour, activity 3 must be reduced in scale. Thus labour used in activity 1 has an opportunity cost in terms of the necessary reduction in activity 3. From Table 2.3 we see that each unit of labour used in activity 3 produces 12·5 bags of grain, so this is the opportunity cost per unit of labour withdrawn from activity 3. The 0·5 units of labour needed for each hectare of activity 1 therefore cost $0{\cdot}5 \times 12{\cdot}5 = 6{\cdot}25$ bags of grain. Since a hectare of activity 1 yields 12 bags of grain, there is a net increase in output of $12 - 6{\cdot}25 = 5{\cdot}75$ bags of grain per hectare of activity 1 introduced. We have shown that it is possible to increase output above 62·5 bags of grain by combining activity 1 with activity 3.

For comparison we now consider the effect of introducing a hectare of activity 2, which again will compete with activity 3 for labour. A hectare of activity 2 yields 16 bags of grain but requires 0·75 units of labour

[4] The reader may note that it is not possible to introduce a whole hectare of activity 1 without competing with activity 3 for land, since there are only 0·5 hectares of land in disposal. This does not matter since all we are concerned with at this stage is whether the introduction of activity 1 is likely to increase or decrease total output. If it will increase output by a given amount per hectare, then the introduction of say half a hectare will increase output by half the amount.

which, at an opportunity cost of 12·5 bags per unit, will cost 0·75 × 12·5 = 9·375 bags of grain. Hence there is a net increase in output of 16 − 9·375 = 6·625 bags of grain per hectare of activity 2 introduced. Since this is greater than the net increase in output of 5·75 bags per hectare of activity 1 introduced, it would appear that the introduction of activity 2 is the more effective way of increasing output.

The question now arises as to how far the substitution of activity 2 for activity 3 can go. We have seen in Table 2.3 that if all 5 units of labour could be used in activity 2 an output of 106·67 bags of grain could be produced, but long before this level of output is reached land becomes an

TABLE 2.4

The technical optimum solution

	Activity 2		Activity 3		Total
Land-use (hectares)	0·8 × 1	+	2·2 × 1	=	3
Labour-use (units)	0·8 × 0·75	+	2·2 × 2	=	5
Total output (bags of grain)	0·8 × 16	+	2·2 × 25	=	67·8

effective constraint. We have also seen that when all the available labour is used on activity 3, 2·5 hectares may be produced so 0·5 hectares of land are left in disposal. At first sight it might appear that only 0·5 hectares of activity 2 can be introduced before there is competition for land, but this is not the case, because the two activities compete for labour. Each hectare of activity 2 requires 0·75 units of labour which is released by reducing activity 3 by 0·75/2 = 0·375 hectares, since each hectare of activity 3 uses 2 units of labour. Hence when a hectare of activity 2 is brought into the plan, only 1 − 0·375 = 0·625 hectares of additional land must be found. Thus the 0·5 hectares of land in disposal may be used to introduce 0·5/0·625 = 0·8 hectares of activity 2. Now, since there are only 3 hectares of land available in total, activity 3 must be reduced to 3 − 0·8 = 2·2 hectares. The new plan or combination of activities consists of 0·8 hectares of activity 2 and 2·2 hectares of activity 3, which will use up all the available land and labour exactly and yield 67·8 bags of grain, as we may see from the calculations in Table 2.4.

If we look for possible ways of increasing the total output still further we find there are none. In other words this is the technically optimum plan, and 67·8 bags of grain is the maximum output possible from the 3 given alternative activities, subject to the 2 given constraints. To check this let us first consider the possibility of including more than 0·8 hectares

of activity 2. Up to this level it does not compete with activity 3 for land, it is simply using up the surplus land which otherwise would be in disposal. Beyond this level, however, it would compete directly with activity 3 for land; each hectare of activity 2 introduced would necessitate a reduction in activity 3 by 1 hectare. As is clear from Table 2.3, activity 3 yields more grain per hectare than does activity 2, so the substitution of the latter for the former activity cannot increase output. Alternatively, if activity 3 is increased beyond 2·2 hectares it competes with activity 2 for labour. Since activity 2 yields more grain per unit of labour than activity 3, this substitution cannot increase total output. Finally, the reader may check that since all the available land and labour are required for the above plan, activity 1 can be introduced only if it is substituted for either or both of the activities already in the plan.

It is clear from Table 2.3 that although activity 1 potentially can produce more grain per unit of labour than either of the other activities, it yields less grain per hectare. But for each hectare of activity 1 introduced at least 1 hectare of one of the other activities would have to be given up in order to release the land. Thus the introduction of activity 1 must cause total output to fall. It is therefore impossible to increase total output beyond 67·8 bags of grain.

The procedure for finding the maximum-product combination of activities when there are fixed constraints, which has just been described, is an exercise—albeit a very simple one—in the planning method known as 'linear programming'. Typically, linear programming is used to solve problems where the objective is to maximize the total returns (or minimize the total costs) of a combination of activities which are subject to a set of fixed constraints and where all relationships are assumed to be linear. Many real problems may be expressed in this form, although the numbers of activities and constraints are usually much larger than in our simple example. Clearly the simple intuitive approach we have used is inadequate for dealing with these larger problems, but mathematical methods of solution and computer programmes have been developed and widely used in real problem solving on farms and elsewhere.[5] Here we are concerned with the underlying theoretical assumptions of the model and the characteristics of the optimum solution, rather than with methods of solution.

When, as in the present example, there are only two constraints, the optimum solution may be found on the isoquant diagram. First we consider the three-dimensional diagram of Fig. 2.8. This surface represents

[5] A selection of the many texts available on linear programming methods is listed at the end of the chapter.

the maximum feasible output levels for combinations of our 3 activities when land is limited to 3 hectares and labour to 5 units. The close similarity between this diagram and Fig. 1.5 should be noted. The only significant difference is that here the surface is made up of flat planes which intersect, whereas in Fig. 1.5 the surface is smoothly curved.

The procedure we have used for finding the maximum output may be traced out on Fig. 2.8. Initially we compared points A, B, and C, which

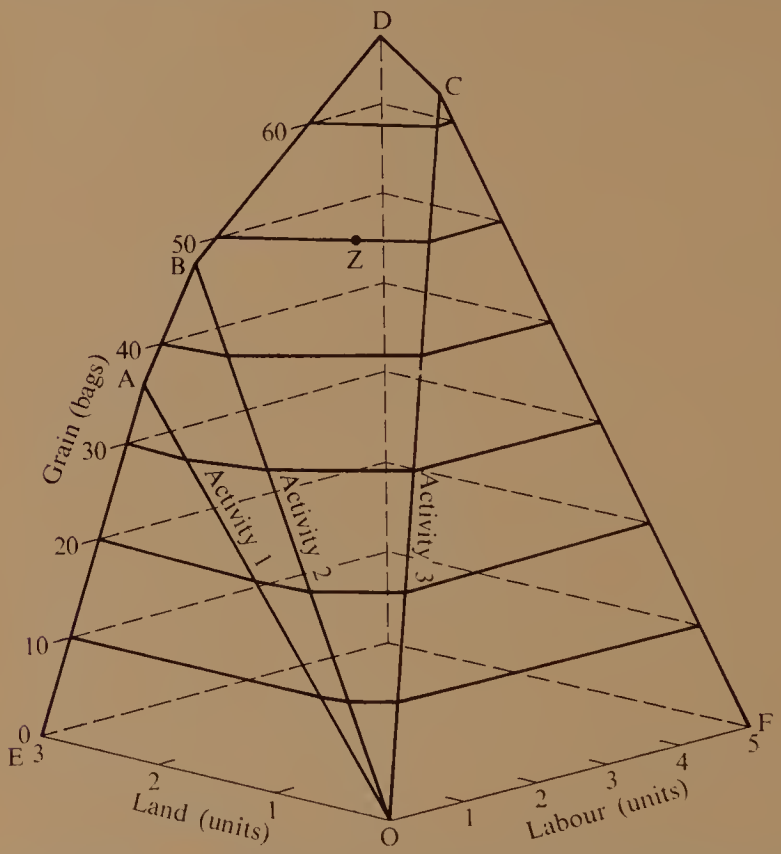


FIG. 2.8. The response surface for two inputs. This is simply Fig. 2.5 in three dimensions with constraints of 3 units of land and 5 units of labour added. Clearly point D, representing a combination of activities 2 and 3, is the point of maximum feasible output.

represent the maximum outputs which can be achieved with each activity on its own. Point C is the highest of these, so activity 3 was chosen and expanded up to the limit set by the labour constraint, where 62·5 bags of grain are produced. Then, by combining activity 2 with activity 3, it was possible to increase output to 67·8 bags of grain at point D where all the available land is in use as well as the available labour. Thus the maximum was found by climbing over the production surface from corner to corner,

increasing the output at each move until the peak was reached. In our search for the maximum output we need to consider only these extreme corner points. Since all relationships are linear and the surface is made up of flat planes, the peak must occur at a corner where lines or planes intersect. The characteristic which distinguishes these corners from all other points on the surface is that at a corner the number of activities in the plan is equal to the number of effective constraints.[6] Thus at point A, there is one activity (1) in the plan and one effective constraint, namely land. At point D there are two activities (2 and 3) in the plan and two

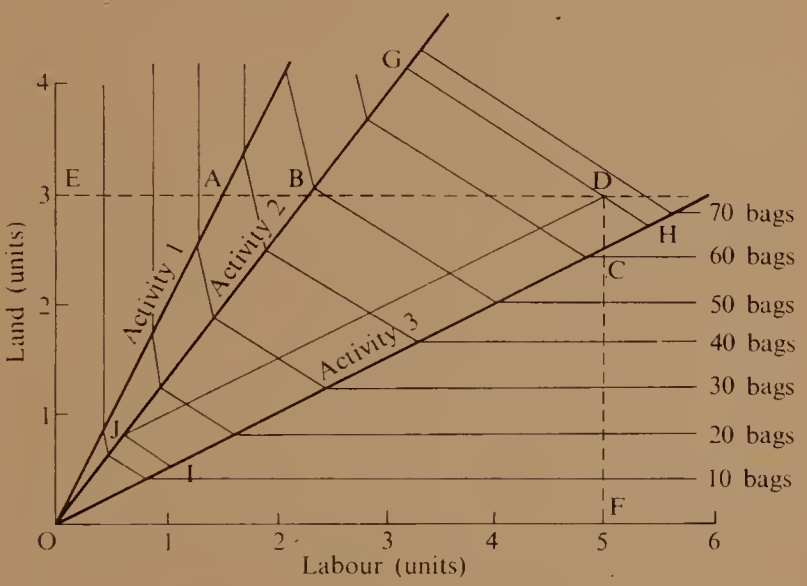


FIG. 2.9. The maximum feasible output. This is Fig. 2.8 in plan view, or Fig. 2.5 with constraint lines added. Maximum feasible output occurs at point D, representing 67·8 bags of grain from 2·2 hectares of activity 3 and 0·8 hectares of activity 2.

effective constraints, both land and labour. In contrast, at point Z, for example, no constraints are effective although there are two activities involved, activity 2 and activity 3.

With the aid of a diagram the point of maximum output is immediately

[6] Exceptions arise, for example:

(a) where two or more factor constraints are in exactly the same proportions as the factor requirements of one activity, in which case the number of effective constraints may exceed the number of activities; or
(b) where one activity has exactly the same input requirements per unit of output, as a combination of other activities, in which case the number of activities could exceed the number of constraints.

obvious and the step-by-step procedure is unnecessary. Fig. 2.9 is a two-dimensional isoquant diagram of the surface shown in Fig. 2.8. It is like Fig. 2.5 but has the constraint lines added.

The vertical constraint line (DF) shows a maximum feasible labour-use of 5 units and the horizontal one (ED) shows a maximum feasible land use of 3 hectares; the highest output that can be achieved is found at point D. It is not possible to use input combinations other than those included in the feasible area, which is the rectangle OEDF. Now, looking at the isoquants spaced at intervals of 10 bags of output it is clear that parts of all of them up to the one for 60 bags of output are included in the feasible area. However, the 70-bag isoquant is entirely outside the feasible area. It is not possible to produce 70 bags of grain in this situation. The line GDH, which just touches the corner point D of the feasible area, is a part of the 67·8-bag isoquant. This output is just feasible with the combination of activities represented by point D.[7]

To complete this analysis, we may consider the changes in the marginal products per unit of each of the two inputs. The effect of increasing land-use on the marginal product per hectare when the labour input is fixed at 3 units can be estimated directly from the linear programme. Thus the first 2·5 hectares were used in activity 3 to yield a marginal product of 25 bags of grain per hectare. When the land-use was increased from 2·5 to 3 hectares the total product rose from 62·5 bags to 67·8 bags, hence the marginal product from half a hectare was 67·8 — 62·5 = 5·3 bags of grain which is 10·6 bags per hectare. The marginal product of land diminishes from an initial 25 bags of grain to 10·6 bags of grain per hectare at the maximum output point.

Now we consider the effect of increasing labour-use with a fixed amount of 3 hectares of land. In effect this implies moving up the surface in Fig. 2.8, or along the straight land-constraint line in Fig. 2.9, from zero labour input through points A and B up to the maximum output at point D. The relationship between labour input and grain output which results is shown in Fig. 2.10. It is extended beyond point D to show the effects of further increases in labour input beyond the 5-unit constraint level. This diagram now closely resembles the response curve of Fig. 1.1 in general shape, although it is angular not smooth.

Between zero labour input and point A, only activity 1 is used yielding

[7] To find the appropriate combination of activities we draw DJ parallel to OH (the ray for activity 3) and JI (parallel to the other isoquants). The combination then consists of OJ = 12·8 bags of grain from activity 2 using 0·8 hectares of land and using 0·6 units of labour, and JD = 55 bags of grain from activity 3 using 2·2 hectares of land and 4·4 units of labour.

24 bags of grain per unit of labour. At point A all 3 hectares of land are devoted to activity 1 so total product is 36 bags of grain (see Table 2.3), and total labour-use is $3 \times 0{\cdot}5 = 1{\cdot}5$ units. Beyond this point, activity 2 is substituted for activity 1, so that more labour can be employed. At point B all 3 hectares of land are used in activity 2 so total product is 48 bags of grain, and total labour-use is $3 \times 0{\cdot}75 = 2{\cdot}25$ units. Thus between point A and point B, labour-use increases by $2{\cdot}25 - 1{\cdot}5 = 0{\cdot}75$ units and output increases by $48 - 36 = 12$ bags of grain. Hence

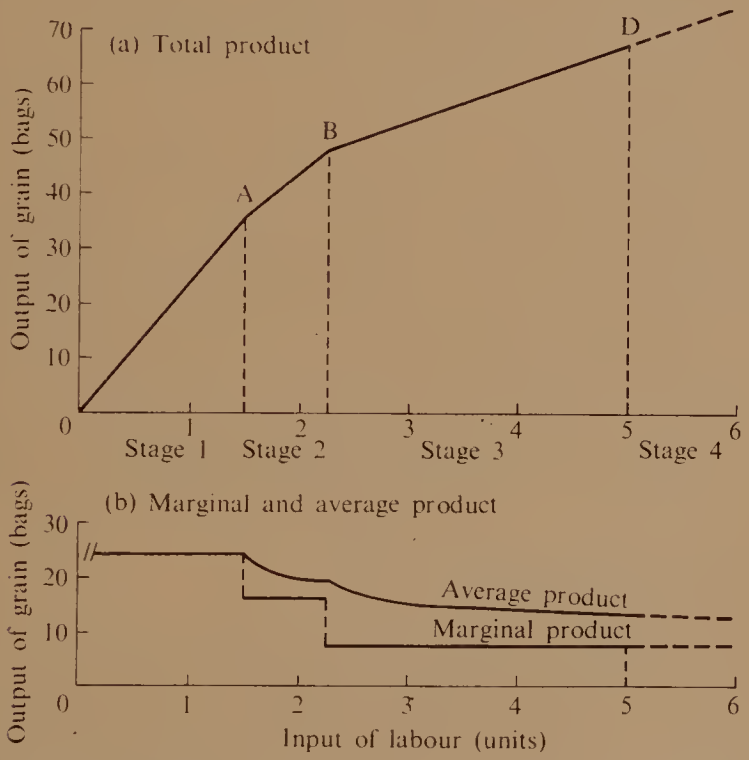


FIG. 2.10. Diminishing returns with linear response. As one activity is substituted for another marginal and average products diminish just as in the case of the smoothly curved production surface.

marginal product per unit of labour is $12/0{\cdot}75 = 16$ bags of grain. Finally, between points B and D output increases by $67{\cdot}8 - 48 = 19{\cdot}8$ bags and labour-use increases by $5 - 2{\cdot}25 = 2{\cdot}75$ units. Hence marginal product per unit of labour is $19{\cdot}8/2{\cdot}75 = 7{\cdot}2$ bags of grain. In fact, additional labour could earn the same marginal product until all 3 hectares of land were devoted to activity 3, requiring 6 units of labour. However, extra labour beyond 6 units would yield zero marginal product.

Thus at D, the point of maximum output, the marginal product per unit of labour is 7·2 bags of grain. The opportunity cost of introducing 1 hectare of activity 1 which requires 0·5 units of labour is $(1 \times 10{\cdot}6) + (0{\cdot}5 \times 7{\cdot}2) = 14{\cdot}2$ bags of grain, which is greater than the output of 12 bags of grain. Thus the introduction of activity 1 would reduce total grain production as we have seen. However, for activity 2 the opportunity cost of $(1 \times 10{\cdot}6) + (0{\cdot}75 \times 7{\cdot}2) = 16$ bags of grain is exactly the same as the output per hectare. Similarly for activity 3 the opportunity cost of $(1 \times 10{\cdot}6) + (2 \times 7{\cdot}2) = 25$ bags of grain is equal to the output per hectare. Neither an increase nor a decrease in the levels of these two activities can bring about an increase in total output.

A further interesting fact may be noted: namely, that total output equals (the marginal product of land multiplied by its quantity) plus (the marginal product of labour multiplied by its quantity). Thus if the supplier of each factor received a reward equal to its marginal physical product, total output would be exactly exhausted. In this example 3 hectares of land at a marginal product of 10·6 bags of grain and 5 units of labour at a marginal product of 7·2 bags of grain exactly account for the total product of 67·8 bags of grain $(3 \times 10{\cdot}6 + 5 \times 7{\cdot}2 = 67{\cdot}8)$. This result is always found when there are constant returns to scale no matter now many inputs are involved (see appendix to Chapter 1, section on returns to scale). It provides a useful check on the accuracy of the calculations of the maximum total product.

Product–product relationships

In order to discuss product–product relationships we now consider alternative activities which yield different products. The analysis is simplified by assuming only two alternative activities one yielding grain and the other roots. The yields per hectare and the labour requirements per hectare both in total and at certain busy periods of the year are given in Table 2.5. From this information we can calculate the input requirement per bag of grain and per tonne of roots. These requirements are given in Table 2.6 together with constraint levels for each of these inputs. Clearly, without constraints there would be no product–product relationship; each one could be expanded independently of others. Only when there are resource constraints are there interrelationships because then different products compete for the limited resources.

These relationships may be illustrated graphically by plotting feasible combinations of grain and root outputs: i.e. by constructing a production-possibility boundary. Consider first the land constraint, which limits

both grain and root production. If all 3 hectares were devoted to the production of grain alone, $3 \times 25 = 75$ bags could be produced. Alternatively if all 3 hectares were devoted to root production, $3 \times 20 = 60$ tonnes could be produced. These two points are plotted and joined by a straight line in Fig. 2.11. This line now joins all combinations of grain and root production which exactly account for 3 hectares of land. For instance, the line passes through the point of 50 bags of grain and

TABLE 2.5

Product output and resource requirements per hectare of grain and roots

		Grain	Roots
Yield	per hectare	25 bags	20 tonnes
Total labour	per hectare (units)	2·0	2·2
June labour	per hectare (units)	0.625	0·20
Dec. labour	per hectare (units)	0	0·60

TABLE 2.6

Resource constraints and requirements per unit of output

	Constraint: total amount available	Grain: input requirement per bag	Roots: input requirement per tonne
Land (hectares)	3	0·04	0·05
Total labour (units)	5	0·08	0·11
June labour (units)	1	0·025	0·01
December labour (units)	0·9	0	0·03

20 tonnes of roots, which as a quick check will show requires $(50 \times 0{\cdot}04) + (20 \times 0{\cdot}05) = 3$ hectares. This line now separates points above it which are infeasible from points below it and within the triangle, which are feasible but which leave some land in disposal.

Straight lines representing the limits on grain and root production set by the other constraints are constructed in a similar way. Thus the amount of each product that could be obtained, if all the labour available in a particular period were devoted to that product alone, is calculated. The estimates, derived from Table 2.6, are set out in Table 2.7. These

values are then plotted on the graph and joined by straight lines as shown in Fig. 2.12. Note that since grain production requires no labour in December this constraint would permit unlimited quantities of grain

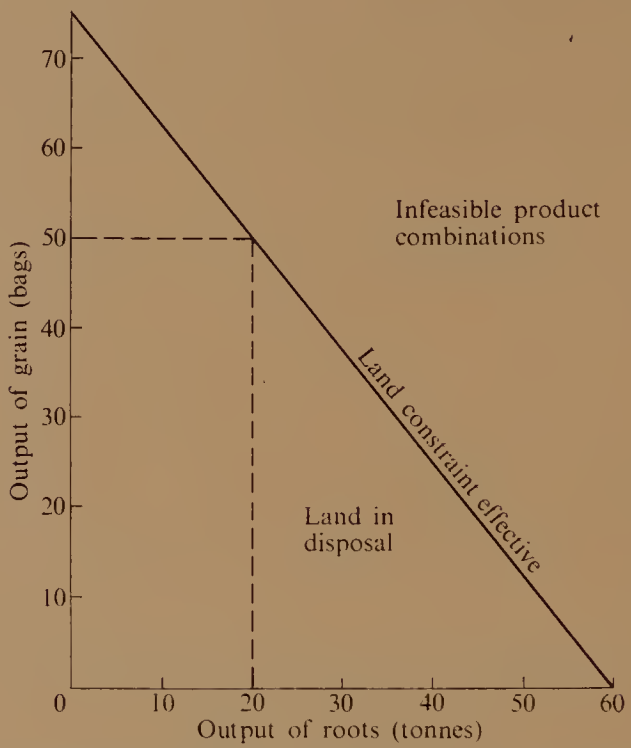


FIG. 2.11. A single constraint and linear product transformation. All feasible combinations of grain and root production are represented by points within the triangle or on its boundary.

TABLE 2.7

Feasible outputs for individual products

	Grain (bags)	Roots (tonnes)
Output from 3 hectares land	75	60
Output from 5 units total labour	62·5	45·7
Output from 1 unit June labour	40	100
Output from 0·9 units December labour	∞	30

to be produced. The constraint line therefore rises vertically from the 'root output' axis and never meets the 'grain output' axis.

The straight lines joining the points O,K,L,M, and N in Fig. 2.12 now trace out the production-possibility boundary which encloses all feasible

combinations of grain and root outputs. The area below and to the left of the graph is clearly infeasible because it represents negative quantities of one or other product.

Point P, for instance, represents 10 tonnes of roots and −5 bags of grain, obviously impossible to produce. Points above and to the right

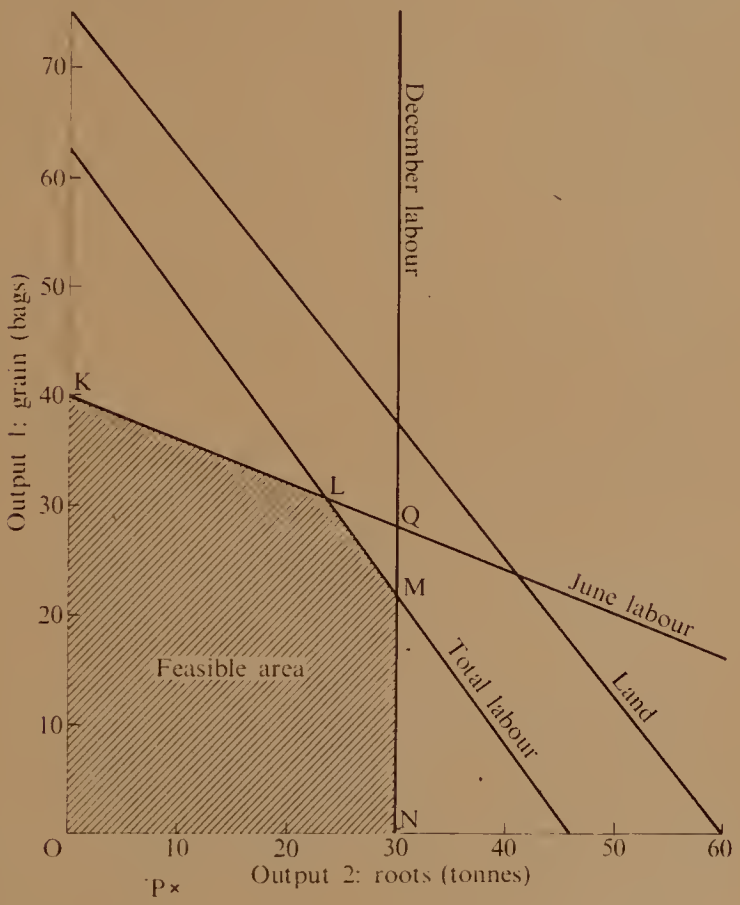


FIG. 2.12. The feasible area. The constraints now limit feasible combinations of grain and root production to the area OKLMN. Note similarity to Fig. 1.7. (p. 24).

of the area OKLMN represent infeasible combinations because certain resource limitations are exceeded. For example, point Q represents a combination of 28 bags of grain and 30 tonnes of roots which is infeasible because the total labour supply of 5 units is inadequate to meet the total requirement of $(28 \times 0{\cdot}08) + (30 \times 0{\cdot}11) = 5{\cdot}54$ units. In other words it implies a negative quantity (− 0·54 units) of total labour in disposal. In practice it might be possible to find the extra

labour when needed so the assumption of a fixed constraint might be a little unrealistic. Thus when solving real problems it is often useful to estimate the effect of varying constraints. However, for the purposes of this discussion we assume they are rigidly fixed.

It is clear from this diagram that if we consider grain production alone, June labour is the effective constraint limiting the maximum feasible output to 40 bags, represented by point K. Of the other constraints 1·4 hectares of land, 1·8 units of total labour, and all the December labour are in disposal.

Now, starting from point K it is clear that if any roots are introduced,

TABLE 2.8

Boundary corners and rates of product transformation

Point in Fig. 2.12	Total product roots (tonnes)	Total product grain (bags)	Rate of product transformation (bags of grain per tonne of roots)
K	0	40	
			0·4
L	23	31	
			1·4
M	30	21	
			∞
N	30	0	

grain production must fall because the two products compete for limited June labour. Each tonne of roots introduced requires 0·01 units of labour, which therefore means a reduction of 0·01/0·025 = 0·4 bags of grain. Thus between point K and point L, the rate of product transformation is 0·4. Between points L and M the products compete for total labour. Each tonne of roots produced requires 0·11 units of labour which is released by a reduction in output of 0·11/0·08 = 1·4 bags of grain. Hence between points L and M the rate of product transformation is 1·4; while between points M and N no further increase in root production is possible and the rate of product transformation is infinite.

The combinations of products at the corner points of Fig. 2.12 are given in Table 2.8, with the rates of product transformation. The rate of product transformation increases and the production-possibility boundary is concave to the origin as in the previous chapter. Now the increasing rate of product transformation is due to the fact that the two products compete for different constraining resources at different

rates. Thus different constraints become effective over different sections of the production-possibility boundary. Between points M and N, grain production is supplementary to root production because the two activities do not compete for the resource December labour. Such supplementary relationships are important in practice. Crops which require labour at different periods of the year are supplementary in the use of labour. Up to a point, the production of one crop may be expanded without reducing the other and without using more labour in total. Intensive pig and poultry enterprises may be introduced without competing with crops for scarce land resources. Many other instances of supplementary enterprises can be found in practice.

The last point to notice about the production-possibility boundary in Fig. 2.12 is that at each of the extreme corner points O,K,L,M, and N the number of real activities at a non-zero level plus the number of resources in disposal is equal to the total number of constraints, namely four.[8] At the origin all 4 resources are in disposal; at point K, 1 activity, grain production, is at non-zero level and 3 resources (land, total labour, and December labour) are in disposal; at points L and M both activities are at non-zero level and 2 resources are in disposal; and finally at point N, 1 activity, root production, is at non-zero level and 3 resources are in disposal.

This finding is consistent with the earlier suggestion that generally the number of real activities at non-zero level equals the number of effective constraints. When a resource is in disposal it means that the constraint is not effective. Thus the number of effective constraints plus the number of resources in disposal must equal the total number of constraints both effective and ineffective.

Each of these extreme points where the number of activities at non-zero level plus the number of resources in disposal is equal to the total number of constraints is known as a 'basic feasible solution'. Such points are important because they define the boundary of the feasible area. It is necessary only to investigate the basic feasible solutions to determine all technically efficient combinations of the given activities.

Summary

The assumptions of (1) a choice of several activities; (2) fixed constraints limiting the choice; and (3) straight-line (linear) relationships result in concave total product lines, convex isoquants, and concave

[8] If we also refer to the non-use of resources as 'disposal activities' then we may say the total number of activities, both real and disposal, is equal to the number of constraints.

production-possibility boundaries as in the case of smooth, continuous relationships. The only real difference is that now the lines are discontinuous, made up of a series of linear segments.

Mathematical appendix

Linear activities and constraints

Consider a productive activity which transforms a single factor into a single product. Let

a = factor input per unit of product output (i.e. the input–output coefficient), which is constant;

b = total amount of the factor available (i.e. the constraint level), also constant; and

X = number of units of output which may be varied subject to the constraint.[9]

Then

$$a \,.\, X \leqslant b \text{ and } X \geqslant 0,$$

i.e. the total factor input must be equal to or less than the total amount of the factor available and negative outputs are impossible. The maximum feasible output X^* is given by

$$X^* = \frac{b}{a}.$$

Note that we define the marginal product per unit of input as $(MP) = 1/a$.

Now, assume two factors are used, both subject to constraints. Let

a_1 = input of factor 1 per unit of product output,

a_2 = input of factor 2 per unit of product output,

b_1 = amount of factor 1 available,

b_2 = amount of factor 2 available.

We now have three constraint inequalities

$$a_1 \,.\, X \leqslant b_1,\ a_2 \,.\, X \leqslant b_2, \text{ and } X \geqslant 0.$$

[9] Note we have now changed the notation. In Chapter 1, X represented the level of input and Y the level of output. However, there we assumed that output is dependent upon the level of input, now we assume that input use depends upon the level of output. In fact the notation we have used in both cases is the conventional notation.

The maximum feasible output is given by

$$X^* = \min\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right),$$

i.e. the minimum value of the maximum feasible outputs subject to the individual constraints or, in other words, the maximum feasible output is determined by the limiting factor or effective constraint.

Let us suppose that factor 1 is the limiting factor. Then:

$$X^* = \frac{b_1}{a_1} \text{ and } X^* < \frac{b_2}{a_2}, \text{ therefore } a_2 \,.\, X^* < b_2.$$

Thus there will be some of factor 2 in disposal, namely $b_2 - a_2 \,.\, X^*$.

Note that if $a_1X < b_1$ and $a_2X < b_2$, $(MP)_1 = 1/a_1$ and $(MP)_2 = 1/a_2$, but if

$$a_1X = b_1 \text{ then } (MP)_2 = 0.$$

When there are 3 activities and 2 constraints we have 6 input–output coefficients; a_{11}, a_{12}, and a_{13} for inputs of factor 1 per unit of activities 1, 2, and 3 respectively; and a_{21}, a_{22}, and a_{23} for inputs of factor 2 per unit of each activity.

The number of units of the three activities are X_1, X_2, and X_3. Since each unit of each activity yields 1 unit of the same product, total product X_t is given by

$$X_t = X_1 + X_2 + X_3. \tag{2.1}$$

The 5 constraint inequalities are

$$a_{11} \,.\, X_1 + a_{12} \,.\, X_2 + a_{13} \,.\, X_3 \leqslant b_1 \tag{2.2}$$
$$a_{21} \,.\, X_1 + a_{22} \,.\, X_2 + a_{23} \,.\, X_3 \leqslant b_2$$

$$X_1 \geqslant 0,\ X_2 \geqslant 0; \text{ and } X_3 \geqslant 0. \tag{2.3}$$

Then

$$(RTS)_{12} \text{ (for activities 1 and 2)} = -\frac{a_{21} - a_{22}}{a_{11} - a_{12}}$$

$$(RTS)_{12} \text{ (for activities 2 and 3)} = -\frac{a_{22} - a_{23}}{a_{12} - a_{13}}.$$

Note however that if there were 3 or more factor constraints, then factor 1 would substitute for 2 or more other factors simultaneously. We would then have a vector (i.e. a row or column of numbers) of substitution rates.

Maximization subject to constraints

If we wish to maximize X_t (to find a technical optimum) it is convenient to convert the inequalities in eqn (2.2) into equalities by introducing so called 'slack variables' or 'disposal activities' X_4 and X_5, representing the non-use of factors 1 and 2.

We then have 2 constraint equations which must be satisfied:

$$\begin{aligned} a_{11} \cdot X_1 + a_{12} \cdot X_2 + a_{13} \cdot X_3 + 1X_4 + 0X_5 &= b_1 \\ a_{21} \cdot X_1 + a_{22} \cdot X_2 + a_{23} \cdot X_3 + 0X_4 + 1X_5 &= b_2 \end{aligned} \tag{2.4}$$

$$\text{and } X_1 \geqslant 0,\ X_2 \geqslant 0,\ X_3 \geqslant 0,\ X_4 \geqslant 0, \text{ and } X_5 \geqslant 0. \tag{2.5}$$

When none of factor 1 is in disposal $X_4 = 0$ so

$$a_{11} \cdot X_1 + a_{12} \cdot X_2 + a_{13} \cdot X_3 = b_1.$$

Similarly when none of factor 2 is in disposal $X_5 = 0$ so

$$a_{21} \cdot X_1 + a_{22} \cdot X_2 + a_{23} \cdot X_3 = b_2.$$

Thus we have 2 constraint equations (2.4) in 5 unknowns: X_1, X_2, X_3, X_4, and X_5. Both these equations must be satisfied, so they represent a set of simultaneous equations. But sets of simultaneous equations where the number of unknown variables exceeds the number of equations generally have an infinite number of solutions (or else no solution at all if the constraint equations are inconsistent or unbounded). Note that any solution which yields negative values for any of the Xs must be rejected as infeasible because of the non-negativity constraints (2.5). We are concerned only with *feasible* solutions.

An important set of solutions may be obtained by reducing the number of unknown variables to equal the number of equations (in this case 2), the remaining variables being given the value of zero. If the reduced system has a solution (i.e. provided the 2 equations in 2 unknowns are independent) it is known as a 'basic solution' and the variables for which values are estimated are known as the basic variables.[10] Basic solutions

[10] The general case, where there are m constraints and n real activities, may be set out in matrix and vector form as $\mathbf{A}.\mathbf{x} = \mathbf{b}$, subject to $\mathbf{x} \geqq 0$, where $\mathbf{A}$ = input–output coefficient matrix of order $m \times (n + m)$ and $\mathbf{x}$ = vector of activity levels, of order $n + m$ allowing for m disposal activities, and $\mathbf{b}$ = vector of constraints, of order m. Any set of m linearly independent columns from $\mathbf{A}$ can form a basis i.e. a square non-singular matrix of order m. The vector $\mathbf{x}$ is partitioned conformably to give $\mathbf{x}_b$ of order m. The corresponding basic solution is then;

$$\mathbf{x}_b = \mathbf{A}_b^{-1}.\mathbf{b}$$

If $\mathbf{x}_b \geqq 0$ it is a basic feasible solution.

which do not violate the non-negativity constraints are known as 'basic feasible solutions' and these represent the corners (or more strictly speaking the extreme points) of the boundary of all feasible solutions.

From eqns (2.4) one basic feasible solution is immediately obvious, namely the one obtained by setting X_1, X_2, and X_3 equal to zero, so that $X_4 = b_1$ and $X_5 = b_2$. This means that nothing would be produced, all of both resources being in disposal. On a graph it would be represented by the origin.[11]. Other basic solutions may be obtained by setting X_2, X_3, and X_4 equal to zero, or X_1, X_4, and X_5 equal to zero, for example. Whether a particular basic solution is feasible or not depends upon the input–output coefficients and the constraint levels.

Example. As an illustration we may consider the simple example set out in Table 2.1, assuming the constraints of 3 hectares of land and 5 units of labour. Since 1 hectare of activity 1 requires $\frac{1}{2}$ a unit of labour and yields 12 bags of grain, 1 bag of grain produced by this activity requires 1/12 hectares of land and 1/24 units of labour. Similarly activity 2 requires 1/16 hectares of land and 3/64 units of labour and activity 3 needs 1/25 hectares of land and 2/25 units of labour per bag of grain. Let

X_1 = number of bags of grain produced by activity 1,

X_2 = number of bags of grain produced by activity 2,

X_3 = number of bags of grain produced by activity 3,

X_4 = number of hectares of land in disposal, and

X_5 = number of units of labour in disposal.

We then have the following pair of constraint equations: (1) for land,

$$1/12\ X_1 + 1/16\ X_2 + 1/25\ X_3 + 1\ X_4 + 0\ X_5 = 3;$$

and (2) for labour,

$$1/24\ X_1 + 3/64\ X_2 + 2/25\ X_3 + 0\ X_4 + 1\ X_5 = 5.$$

All the basic solutions (there are 10 in all) for this pair of equations are given in Table 2.9.

As we have seen, the basic feasible solutions represent the extreme corner points of the boundary of all feasible solutions. Solutions inside the boundary, though feasible, are technically inefficient in that a greater total product could be obtained by expanding the activities up to the boundary (i.e. up to the limit set by the constraints). Thus all

[11] Clearly such a solution would not be feasible if there were constraints requiring that not less than a certain output should be obtained.

TABLE 2.9

Basic solutions for three real activities and two constraints

	X_1 (bags)	X_2 (bags)	X_3 (bags)	X_4 (hectares)	X_5 (units)	Comment	Total product (bags)
1	0	0	0	**3**	**5**	The origin in Figs 2.8 and 2.9	0
2	**36** (3)	0	0	0	**3·5**	Point A in Figs 2.8 and 2.9	36
3	0	**48** (3)	0	0	**2·75**	Point B in Figs 2.8 and 2.9	48
4	0	0	**62·5** (2.5)	**0·5**	0	Point C in Figs 2.8 and 2.9	62·5
5	**8** (0·67)	0	**58·33** (2·33)	0	0	Combination of activities 1 and 3 not shown in Figs 2.8 and 2.9	66·33
6	0	**12·8** (0·8)	**55** (2·2)	0	0	Point D in Figs 2.8 and 2.9	67·8
7	**120** (10)	0	0	**−7**	0	Infeasible; land disposal negative	
8	0	**107·2** (6·7)	0	**−3·7**	0	Infeasible; land disposal negative	
9	0	0	**75** (3)	0	**−1**	Infeasible; labour disposal negative	
10	**−132** (−11)	**224** (14)	0	0	0	Infeasible; activity 1 negative	

Note. The number of hectares of each real activity appearing in a basic solution is given in parentheses.

technically efficient points must lie on the boundary, either at or between extreme corner points. Since the boundary is made up of straight-line segments, we need estimate only the extreme points (i.e. determine the basic feasible solutions) to define the whole boundary. Furthermore if we are interested in the point of over-all maximum output, it will occur at one (or more) of these extreme points (basic feasible solutions). Thus the point of maximum output may be found by calculating all basic feasible solutions and selecting the one (or more) that yields the highest total output.

In real problems, with more than 3 real activities and 2 constraints there are more than 10 basic solutions. Indeed, the number of basic solutions increases rapidly as the numbers of activities and constraints increase.[12] Thus complete enumeration and investigation of all basic solutions is *not* the most efficient method of finding the point of maximum output. Techniques of linear programming are step-by-step procedures for finding a maximum (or minimum) which ensure that at each step a movement is made to a basic feasible solution which yields a higher return (or lower cost) than the previous one. This feature ensures that many relatively unprofitable basic feasible solutions are ignored in the search for the maximum.

The revised simplex method

One such technique of linear programming is the 'revised simplex method', which may be summarized as follows.

1. Select an initial basic feasible solution (BFS). Any BFS will do, though if the origin represents a feasible solution it is convenient to start here. At the origin only disposal activities are in the basis and the total product is zero.

2. Calculate (*MP*) for the factors of production as implied by the current BFS. So long as a resource constraint is not effective (*MP*) for that resource is zero. Thus at the origin the (*MP*)s for all factors are zero. For other BFSs there is at least one real activity at a positive level and at least one effective constraint. Now, to calculate the marginal products of the limiting factors (those for which the constraints are

[12] If there are n real activities and m constraints, hence m disposal activities, the potential number of basic solutions is given by the formula:

$$\frac{(n+m)!}{m!n!},$$

where $m! = (m-1)(m-2)\ldots 3.2.1$. However, some bases do not have solutions so the actual number of basic solutions may be less.

effective) we may use the fact that under constant returns to scale, total product exactly equals the sum of the factor inputs multiplied by their (*MP*)s for each activity. For example if real activity number 2 is in the BFS and there are only two factors of production with (*MP*)s w_1 and w_2 we have the following equation:

$$a_{12} \cdot w_1 + a_{22} \cdot w_2 = 1. \tag{2.6}$$

With only one real activity in the BFS, only one factor will have a positive (*MP*), namely the factor which is completely exhausted and hence not in disposal. Its (*MP*) will be the reciprocal of the relevant input–output coefficient. Thus if factor 1 is in disposal then eqn (2.6) becomes;

$$a_{12} \cdot 0 + a_{22} \cdot w_2 = 1.$$

Therefore

$$w_2 = 1/a_{22}.$$

With more than one real activity in the BFS and hence more than one effective constraint the (*MP*)s may be found by solving the relevant set of simultaneous equations. For example, with 2 limiting constraints and real activities 2 and 3 we have the 2 equations:

$$a_{12} \cdot w_1 + a_{22} \cdot w_2 = 1$$
$$\text{and } a_{13} \cdot w_1 + a_{23} \cdot w_2 = 1.$$

Note that the number of equations is normally equal to the number of unknown *w*s because the number of real activities in a BFS is normally equal to the number of effective constraints.

3. Calculate the net gain of each of the activities *excluded* from the current BFS, using the estimated marginal product per unit of each factor as the opportunity cost.

For instance, let us assume that activity 1 is excluded from the BFS when there are 2 effective constraints. We then have:

Net gain from introducing 1 unit of activity 1

$$= 1 - a_{11} \cdot w_1 - a_{21} \cdot w_2.$$

If the net gain is negative or zero for all the excluded activities, the current BFS represents the maximum-output combination and the analysis is complete. When the net gains for all excluded activities are negative the maximum-output combination is unique, but this is not the case if any are zero.

If the net gain is positive for any of the excluded activities then total output can be increased by introducing one of these activities. It does not

matter which one is chosen, though commonly the one with the highest net gain is selected.

4. Since the number of activities in the basis equals the number of constraint equations, the introduction of a new activity must be accompanied by the elimination of one of the activities in the current basis. The one which is eliminated is the one which effectively limits the level of the new incoming activity. At the origin this is easy since we have:

$$X = \min \left(\frac{b_1}{a_1}, \frac{b_2}{a_2} \text{ etc.} \right)$$

for the incoming activity. However, at any other basis we must use the more general rule that the level of the incoming activity is limited by the level of one of the activities already in the basis. Let us assume that X_2 and X_3 are the levels of the 2 activities already in the basis and X_1 is the level of the incoming activity, then:

$$X_1 = \min \left(\frac{X_2}{p_{21}}, \frac{X_3}{p_{31}} \right),$$

where p_{21} and p_{31} are the rates at which the new activity substitutes for the activities in the current basis. These rates of substitution are measured along the boundary of the feasible area and not along the isoquants so they differ from the (*RTS*)s defined earlier. They are estimated by solving a set of simultaneous equations, relating the original input–output coefficients of the incoming activity to the original input–output coefficients of the activities in the basis. For example p_{21} and p_{31} may be estimated by solving[13]

$$a_{12} \cdot p_{21} + a_{13} \cdot p_{31} = a_{11}$$

$$a_{22} \cdot p_{21} + a_{23} \cdot p_{31} = a_{21}.$$

The current values of X_2 and X_3 are given by the BFS so the minimum ratio can be determined and this identifies the limiting basic activity which is the one to be eliminated.

5. Find the new BFS incorporating the new incoming activity and excluding the activity to be eliminated. This is the end of one step or

[13] Using the notation of footnote 10 and letting $\mathbf{a}^j$ = the jth column of $\mathbf{A}$ not in $\mathbf{A}_b$, the corresponding rates of substitution for the current basis $\mathbf{p}^j$ are given by

$$\mathbf{p}^j = \mathbf{A}_b^{-1} \cdot \mathbf{a}^j.$$

iteration. The procedure may now be repeated starting at stage 2, as many times as are needed to reach the point of maximum output.[14]

Example. To illustrate this procedure we return to the example given above.

1. The initial BFS representing the origin consists of 3 hectares of land and 5 units of labour in disposal (BFS no. 1 in Table 2.9).
2. At this stage (MP) of land, $w_1 = (MP)$ of labour, $w_2 = 0$.
3. The net gain per unit of each of the real activities 1, 2, and 3 is therefore 1 bag of grain (minus zero opportunity cost). Therefore total output can be increased by introducing any one of the three.

We choose to introduce activity 3.

4. The maximum feasible level of activity 3 is given by:

$$X_3 = \min\left(\frac{3}{1/25}, \frac{5}{2/25}\right)$$

$$= \min\begin{matrix}(75 & , 62\tfrac{1}{2}).\\ \text{land} & , \text{labour}\end{matrix}$$

[14] If in addition to the conditions set out in footnote 10 we introduce a vector $\mathbf{c}$ of return per unit of each activity, of order $n + m$, and an objective function:

$$\text{maximize } Z = \mathbf{c} \,.\, \mathbf{x},$$

we have a typical linear programming problem. (n.b. In the present example the elements of $\mathbf{c}$ are unity for real activities and zero for disposal activities.)

If we write $\mathbf{A_b}$, $\mathbf{x_b}$, and $\mathbf{c_b}$ for the coefficients, activity levels, and unit returns for activities in the basis and $\mathbf{A_0}$, $\mathbf{x_0}$ and $\mathbf{c_0}$ as the corresponding values for activities not in the basis, the revised simplex method may be summarized as follows.

1. Choose any feasible basis and solve

$$\mathbf{x_b} = \mathbf{A_b}^{-1} \,.\, \mathbf{b} \quad \text{and} \quad Z = \mathbf{c_b} \,.\, \mathbf{x_b}.$$

2. Find shadow prices (vector $\mathbf{y}$ of order m) as

$$\mathbf{y} = \mathbf{c_b} \,.\, \mathbf{A_b}^{-1}.$$

3. Use shadow prices to calculate the marginal net gain π_0 from introducing non-basis variables into the basis

$$\pi_0 = \mathbf{c_0} - \mathbf{y} \,.\, \mathbf{A_0}$$

If all elements of π_0 are non-positive the optimum solution has been found. If not, choose the variable with the largest positive value, the jth, as the incoming variable.

4. For the jth variable calculate $\mathbf{p}^j$ as

$$\mathbf{p}^j = \mathbf{A_b}^{-1} \,.\, \mathbf{a}^j$$

(as in footnote 13). The outgoing variable is then given by minimum $(x_i/p_i{}^j), x_i \in \mathbf{x_b}$. This ensures that no x_i is negative in the new basis.

5. Substitute the incoming variable for the outgoing in the basis and solve as in step 1.

Thus X_3 will equal $62\frac{1}{2}$ in the new basis and labour disposal is the activity to be eliminated.

5. The new BFS includes $62\frac{1}{2}$ units of activity 3 and $\frac{1}{2}$ hectare of land disposal (BFS no. 4 in Table 2.9). *End of first iteration*

2. We find the (*MP*)s of the factors of production by solving

$$1 \,.\, w_1 + 0 \,.\, w_2 = 0$$

for land disposal, and

$$1/25 \,.\, w_1 + 2/25 \,.\, w_2 = 1$$

for activity 3, which yields the solution

$$w_1 \,((MP) \text{ of land}) = 0$$

(as we expect for a factor in disposal);

$$w_2 \,((MP) \text{ of labour}) = 12\tfrac{1}{2} \text{ bags of grain.}$$

3. The net gain per unit of activity 1 is therefore

$$1 - 1/12 \,.\, w_1 - 1/24 \,.\, w_2 = 1 - 0 - 25/48 = + 23/48 \text{ bags}$$

and the net gain per unit of activity 2 is

$$1 - 1/16 \,.\, w_1 - 3/64 \,.\, w_2 = 1 - 0 - 75/128 = + 53/128 \text{ bags.}$$

Hence total product can be increased by introducing either activity. We choose activity 1 since this yields the higher net gain.

4. The rates of substitution p_{31} and p_{41} are found by solving

$$1/25 \,.\, p_{31} + 1 \,.\, p_{41} = 1/12,$$
$$2/25 \,.\, p_{31} + 0 \,.\, p_{41} = 1/24,$$

which gives the result $p_{31} = 25/48$ and $p_{41} = 1/16$.
Hence

$$X_1 = \min \left(\frac{62\frac{1}{2}}{25/48}, \frac{\frac{1}{2}}{1/16} \right)$$
$$= \min (120 \quad , 8)$$
$$\text{activity 3} \quad , \text{ land disposal}$$

Thus land disposal is the activity to be eliminated.

5. The new BFS includes 8 units of real activity 1 and $58\frac{1}{3}$ units of activity 3 so the total product is $66\frac{1}{3}$ bags of grain (BFS no. 5 in Table 2.9). *End of second iteration*

2. We find (*MP*)s of the factors of production by solving

$$1/12 \,.\, w_1 + 1/24 \,.\, w_2 = 1 \text{ for activity 1}$$
$$1/25 \,.\, w_1 + 2/25 \,.\, w_2 = 1 \text{ for activity 3,}$$

which yields the solution:

$$w_1 \text{ ((}MP\text{) of land)} = 23/3$$
$$w_2 \text{ ((}MP\text{) of labour)} = 26/3.$$

3. The net gain per unit of activity 2 is therefore

$$1 - (1/16 \,.\, 23/3) - (3/64 \,.\, 26/3) = 11/96 \text{ bags.}$$

Hence we introduce activity 2.

4. The rates of substitution p_{12} and p_{32} are found by solving

$$1/12 \,.\, p_{12} + 1/25 \,.\, p_{32} = 1/16$$

and

$$1/24 \,.\, p_{12} + 2/25 \,.\, p_{32} = 3/64,$$

which gives the result $p_{12} = 5/8$ and $p_{32} = 25/96$.

Hence

$$X_2 = \min \left(\frac{8}{5/8}, \frac{66\frac{1}{3}}{25/96} \right)$$
$$= \min (12\tfrac{4}{5}, 254\tfrac{18}{25})$$
$$\text{activity 1, activity 3}$$

Thus activity 1 is to be eliminated.

5. The new BFS includes $12\frac{4}{5}$ units of activity 2 and 55 units of activity 3 so total product is $67\frac{4}{5}$ bags of grain (BFS no. 6 in Table 2.9). *End of third iteration*

2. Now we have

$$\tfrac{1}{16}w_1 + \tfrac{3}{64}w_2 = 1, \text{ for activity 2}$$
$$\tfrac{1}{25}w_1 + \tfrac{2}{25}w_2 = 1, \text{ for activity 3,}$$

which has the unique solution

$$w_1 = 10\tfrac{3}{5} \text{ and } w_2 = 7\tfrac{1}{5}.$$

3. This result implies that the net gain per unit of each of the activities excluded from this BFS is

for land: $-10\frac{3}{5}$ bags of grain (disposal activity);

for labour: $-7\frac{1}{5}$ bags of grain (disposal activity);
and for activity 1, $1 - (\frac{1}{12} \cdot 10\frac{3}{5}) - (\frac{1}{24} \cdot 7\frac{1}{5}) = -\frac{11}{60}$.

Hence it is not possible to increase total output and we have found the point of maximum output.

Alternative products

If different activities yield different products but are subject to the same set of constraints, the feasible combinations of products may still

TABLE 2.10

Basic solutions for 2 products and 4 constraints

	X_1	X_2	X_3	X_4	X_5	X_6	Comment
1	0	0	3	5	1	0·9	The origin in Fig. 2.12
2	40	0	1·4	1·8	0	0·9	Point K in Fig. 2.12
3	30·8	23	0·62	0	0	0·21	Point L
4	21	30	0·65	0	0·175	0	Point M
5	0	30	1·5	1·7	0·7	0	Point N
6	0	100	−2	−6	0	−2·1	Infeasible
7	0	45·5	0·73	0	0·5	−0·47	Infeasible
8	0	60	0	−1·6	0·4	−0·9	Infeasible
9	∞	0	− ∞	− ∞	− ∞	0	Infeasible (non-existent)
10	62·5	0	0·5	0	−0·56	0·9	Infeasible
11	75	0	0	−1	−0·88	0·9	Infeasible
12	37·5	30	0	−1·3	−0·24	0	Infeasible
13	23·5	41	0	−1·4	0	−0·33	Infeasible
14	200	−100	0	0	−3	12	Infeasible
15	28	30	0·38	−0·54	0	0	Infeasible, point Q in Fig. 2.12

be expressed by equations such as (2.4) and (2.5) above and defined by the set of basic feasible solutions.

For example, we may express the constraints set out in Table 2.6 by the following system of equations and inequalities:

$$0{\cdot}04\,X_1 + 0{\cdot}05\,X_2 + 1\,X_3 + 0\,X_4 + 0\,X_5 + 0\,X_6 = 3,$$
$$0{\cdot}08\,X_1 + 0{\cdot}11\,X_2 + 0\,X_3 + 1\,X_4 + 0\,X_5 + 0\,X_6 = 5,$$
$$0{\cdot}025\,X_1 + 0{\cdot}01\,X_2 + 0\,X_3 + 0\,X_3 + 1\,X_5 + 0\,X_6 = 1,$$
$$0\,X_1 + 0{\cdot}03\,X_2 + 0\,X_3 + 0\,X_4 + 0\,X_5 + 1\,X_6 = 0{\cdot}9,$$

and $X_1, X_2, X_3\, X_4\, X_5, X_6 \geqslant 0$,

where X_1 = bags of grain production, X_2 = tonnes of root production, X_3 = hectares of land in disposal, X_4 = units of total labour in disposal, X_5 = units of June labour in disposal, and X_6 = units of December labour in disposal. All the basic solutions are set out in Table 2.10.

Where there are only two real activities and hence two products (RPT) is a simple ratio of the relevant input–output coefficients. For instance over the interval where constraint 1 is effective (i.e. the disposal activity for factor 1 is at zero level)

$$(RPT)_{21} = \frac{a_{12}}{a_{11}}.$$

Similarly, over the interval where constraint 2 is effective

$$(RPT)_{21} = \frac{a_{22}}{a_{21}}.$$

However, with more than 2 real activities, then 1 product will be transformed into several if not all the other products as we move from 1 BFS to another. Hence there is a vector of (RPT)s for each product.

Suggestions for further reading

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3. Economic optimization

Economic rationality

The technical relationships between inputs and outputs, whether they be curved or linear, set a limit on the types and quantities of crops and livestock which can be produced with a given combination of resources. All of the specific technical relationships the farmer could make use of, together constitute the available technology. Generally speaking the available technology has expanded over time and presumably will continue to do so but for any individual or society at a given point in time there is a limited stock of known technology.

At the same time the alternatives open to the farmer in his productive activity may be constrained by limitations on physical resources, by basic survival needs and perhaps by social customs and taboos. The available technology together with these constraints determines the set of feasible alternatives open to the farmer. The wider are the technological and the resource bases, and the greater the freedom from social constraints, the wider will be the farmer's range of choice. Very poor producers working with a primitive technology may have very little freedom of choice but in most situations the farmer is left with a wide range of choice as to what to produce, how much to produce and what methods to use.

In order to analyse these choices we now assume 'economic rationality' on the part of the farmer, which means that he chooses among the feasible alternatives in such a way as to maximize his satisfaction. This in turn implies that he is fully aware of the opportunities open to him and is capable of evaluating them, or at least placing them in some order of preference. For example he must know exactly how much milk he could produce with given quantities of land, labour, and capital; how much wheat he could produce with the same resources; and he must be able to decide which he would prefer to have, the milk, the wheat, or some of the resources available for other uses. Thus we are assuming that the farmer is completely informed and there is no uncertainty. Later on, in the next chapter this assumption will be relaxed somewhat, but for the present the simplifying assumption of perfect knowledge is made.

If we make a further simplifying assumption that all inputs are bought or hired and all outputs are sold, then prices may be used to evaluate

satisfactions. Revenues from sales represent additions to, and costs of inputs subtractions from, total satisfaction. The difference between total revenue and total cost, which we shall call 'profit', represents the net addition to total satisfaction. The farmer then maximizes his satisfaction by maximizing his profit.

Note that, strictly speaking, the satisfactions are derived from the consumption of goods and services that may be purchased out of profit. Thus where trade is possible the entire set of feasible alternatives includes both production and exchange activities. The farmer may convert his available resources into consumption goods either by production or by exchange or both. However, to simplify the analysis it is convenient to separate production decisions from consumption choices. This may be achieved by treating prices as measures of relative value. Here we use prices to allow us to concentrate on production decisions. However, in following this approach it should be remembered that the real costs of inputs are the opportunity costs. The cash outlay on a variable input is not in itself the cost, but merely a measure of the value of other opportunities forgone. The concept of opportunity cost is generally applicable to all inputs whether or not they are purchased. The market price of a purchased input is just a particular case of an opportunity cost.

Now it is most unlikely that any farmer makes all his decisions with the sole objective of maximizing profits. Apart from the aim of avoiding risk, which will be discussed in the next chapter, farmers have other objectives besides profit making. Some want a quiet and uncomplicated life; others want prestige and positions of authority in their local community. The satisfactions derived from the affection and respect of one's family or from living and working in the country do not appear in the calculation of farm profit. Indeed some choices are made without any clear objective, when they are dictated by habit or tradition. Thus one might well ask whether there is any justification for the assumption that farmers maximize profits.

However, this is again a theoretical model of reality intended as a useful simplification and not as a complete explanation of the choices farmers make. The model may be used either for *prediction* of what farmers are likely to do in general under given conditions, or for *prescription* of what they ought to do.[1] Thus the assumption that they maximize profits enables us to predict that if the price of a particular

[1] Predictions and factual statements are sometimes called *positive* statements in contrast with prescriptions and value judgements which are called *normative* statements.

product, say pigmeat, rises in relation to other prices, farmers will produce more pigs. Similarly if wages rise in relation to the cost of machinery we can predict that machinery will be substituted for labour. These predictions can be tested by comparison with the way in which farmers actually do respond to price changes. If the assumption of profit maximization is wildly unrealistic the predictions will be refuted by the evidence. In fact most empirical studies have shown that farmers do respond to price changes in the way predicted by the model.[2] Most of the important theories of economics are based on the assumption that producers try to maximize profits.

Alternatively we may use our analysis to prescribe what farmers ought to do in order to maximize profits. This is the approach taken in much farm management advisory work.[3] It is taken for granted that profit maximization is the ultimate objective, either because it is rational behaviour or because a farmer who fails to make profits will be unable to survive. He must eventually go bankrupt. The combination of inputs and outputs which maximizes profit is known as the 'economic optimum'. Combinations of inputs and outputs differing from the economic optimum are sometimes called 'economically inefficient'. If the costs and revenues of alternative policies can be calculated, the farmer can be told the cost of economic inefficiency in terms of the profit forgone. A farmer who prefers a particular breed of cows for their appearance rather than for their milking ability might change his mind if he knew how much profit he was loosing by keeping them. Nevertheless a good farm-management advisor recognizes that profit making is not the only aim in farming and tempers his advice accordingly.

The economic optimum for a single variable input

One further assumption will be made for the purposes of this analysis, namely that the prices per unit of variable inputs and product outputs are fixed and constant regardless of how much the individual farmer produces. This is not an unreasonable assumption since in most countries the agricultural industry is made up of very many farms. Where prices of agricultural inputs and outputs are determined by the operation of unrestricted market forces, the quantities of inputs used and of outputs sold by an individual farmer are unlikely to have a significant effect on prices. We may refer to such conditions as 'pure competition'.

[2] For some examples see Cowling, K., Metcalf, D., and Rayner, A. J. (1970). *Resource structure of agriculture: an economic analysis*. Pergamon Press, Oxford.

[3] See for instance Dexter, K. and Barber, D. (1961). *Farming for profits*. Penguin Books, Harmondsworth.

Alternatively, if the government intervenes to fix agricultural prices it is unlikely to be influenced in its decisions by the activities of an individual farmer. Thus prices will be treated as constants outside the farmer's sphere of influence. His decisions are limited to the choice of quantities of inputs and outputs given the prevailing market prices.

The economic optimum level of use of a single variable input may be illustrated using the hypothetical production-response data given in

TABLE 3.1

Product values and factor costs

1	2 Costs			3	4	5	6
Resource input (units)	Fixed costs (£)	Variable factor cost (£)	Total cost (£)	Value of total product (£)	Profit (£)	Value of marginal product (£)	Unit factor cost (£)
0	10	0	10	0	−10		
						7	5
1	10	5	15	7	−8		
						13	5
2	10	10	20	20	0		
						11	5
3	10	15	25	31	6		
						6	5
4	10	20	30	37	7		
						3	5
5	10	25	35	40	5		

Notes

Derived from Table 1.2. Product price = £1 per bag, unit factor cost = £5, and fixed cost = £10.

Profit is maximized where the value of the marginal product is most nearly equal to unit factor cost; i.e. at four units of resource input.

Table 1.2 (p. 9) and the assumed prices of £1 per bag of grain, £5 per unit of labour, and a fixed cost of £10. The resulting revenues and costs are given in Table 3.1.

The total profit for each level of labour input is given in column 4 of the table. It should be noted, for less than 2 units of labour input, profits are negative, in other words grain production at these levels of labour input would result in losses. At 2 units of labour input profit is exactly zero but above this level positive profits are made. The highest profit shown in the table is £7 which is obtained when 4 units of labour are used. This level of labour input is therefore the economic optimum.

An alternative, and more useful, approach to finding the economic optimum is to compare the extra cost of each additional unit of labour with the extra revenue it will earn. So long as the extra revenue exceeds extra cost, profit is increased by using more of the variable input. Since prices are constant, the cost of every unit of input is the same and is known as the unit factor cost. In this example, where labour is the variable input, the wage rate of £5 per unit is the unit factor cost, given in column 6 of the table. The extra revenue is measured by multiplying the marginal physical product per unit of input by the product price to give the money value of the marginal product as in column 5 of the table. If we now compare columns 5 and 6 it is clear that for labour inputs up to and including the fourth unit the value of the marginal product exceeds the unit factor cost, so profit is increased by each additional unit of labour. At levels of input above 4 units, profit is decreased because the value of the marginal product is less then the unit factor cost.

Here, then, we have the basic principle for deciding how much to use of a variable input; to expand its use so long as the value of the marginal product exceeds the unit factor cost. The precise economic optimum is then reached at the point where the two are equal, although where, as in the table, inputs are varied by relatively large steps they may not be exactly equal.[4]

The precise point of economic optimum can be found by using the response curve and marginal and average product curves from Fig. 1.1 (p. 10), together with lines showing the costs of the inputs as in Fig. 3.1. It should be noted that the product curves are identical with those in Fig. 1.1; we merely alter the vertical scale so that the product is measured in terms of value rather than in physical units. In Fig. 3.1(a) the total cost, made up of the fixed cost of £10 and an increasing variable cost, is represented by the straight line CDE. The difference between total value product and total cost, BD, is profit. It is at a maximum of £7.5 at 3·7 units of labour input where the slope of the total value product curve (which is the value of the marginal product) is equal to the slope of the

[4] Alternatively the analysis can be made in terms of bags of grain. The cost of labour measured in terms of grain is given by the inverse price ratio; i.e. the wage rate divided by the price per bag of grain. The economic optimum is then defined as the point where the marginal *physical* product equals the inverse price ratio. Thus, at the economic optimum, labour receives a wage equal to its marginal product in bags of grain.

Note that these rules apply only when prices are constant. To cover cases where prices of inputs and outputs vary the economic optimum may be defined as the point where marginal value product equals marginal factor cost.

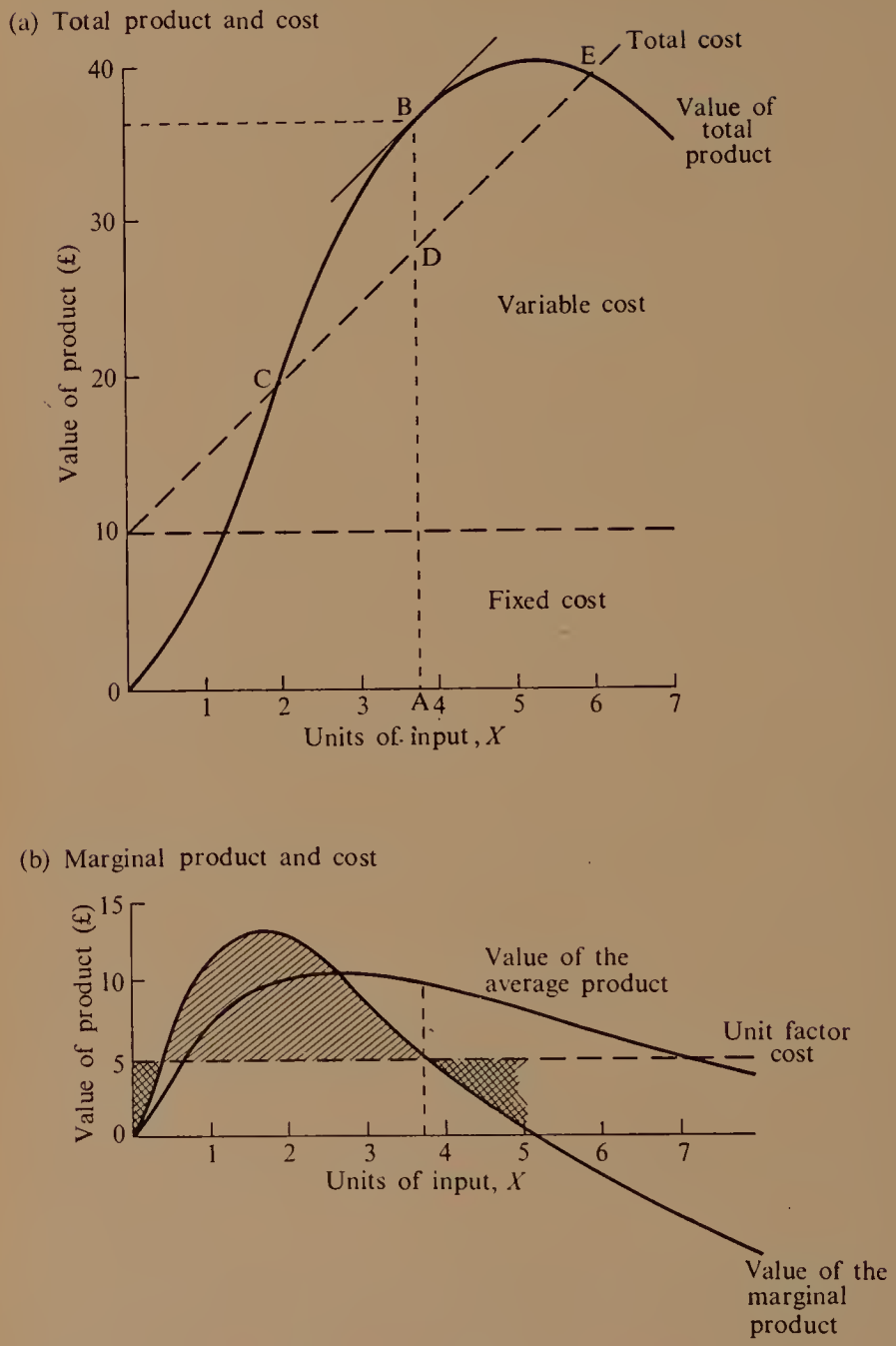


FIG. 3.1. The economic optimum for a single variable factor. Costs are now represented on the production-response curves of Fig. 1.1 (p. 10), with the product now measured in value terms grain being valued at £1 per bag. Profit (the difference between the value of the product and the total cost) is maximized at OA (about 3·7) units of input.

total cost line (which is the unit factor cost). Note that the slope of the total cost line would be unaltered by changes in the level of fixed costs, so the economic optimum is unaffected, unless fixed costs rise so high that production becomes uneconomic.

On the graph of the value of the marginal product (Fig. 3.1(b)), unit factor cost is the same for each extra unit of labour so it is represented by a horizontal straight line. The shaded area minus the cross-hatched areas represents the surplus out of which fixed costs must be paid to leave the profit. Since the fixed costs are constant an increase in this area must represent an increase in profit. The area, and hence profit, is maximized at 3·7 units of labour input.

Of course it is always possible that the most profitable level of output will be less profitable than not producing at all. Some productive activities can only yield losses. To cover this possibility we must add the condition that nothing will be produced unless the value of the average product exceeds the unit factor cost. This implies that where the average product is at a maximum (at which point it is equal to marginal product) the value of both average and marginal product will be greater than the unit factor cost. From this point onwards the law of diminishing returns applies; as the variable input is increased so the value of the marginal product per unit falls until the level of the unit factor cost is reached. Thus the economic optimum can exist only where there are diminishing marginal returns. If marginal products did not decline, farmers could increase their profits indefinitely by using more and more labour.

The economic optimum is sometimes called the equilibrium level of input, implying that it is a point of balance. Any change in relative prices or in the response curve alters the optimum. For instance if the product price rises in relation to the factor price the economic optimum occurs at a higher level of input and output. In the present example, if the price of grain doubled in relation to the price of labour the economic optimum level of labour input would be increased to 4·4 units, as the reader may confirm by drawing appropriate cost lines in Fig. 3.1. If the wage rate fell by half while the price of grain remained constant, the effect would be identical. Note that if the prices of both the variable input and the product move together so that their ratio stays constant, the economic optimum is unchanged.

New techniques of production may alter the production function so that marginal products per unit of labour are increased. If prices are unaltered by the change, the economic optimum level of labour-use is increased. The economic optimum level of output is increased in two ways, first because the marginal and average product of each unit of

labour is increased and secondly because more units of labour are used. However, if the new techniques are widely adopted so that aggregate output is increased significantly, it is most unlikely that the product

TABLE 3.2

Least-cost combinations (from Table 1.4)
Price of labour = £5 per unit; price of land = £10 per unit; inverse price ratio = 5/10 = 0·5

Labour (units)	Land (units)	Rate of technical substitution	Total cost (£)
(a) For 40 bags of grain output			
1	7·0		75
		4·2	
2	2·8		38
		1·0	
3	1·8		33
		0·5	
4	1·3		33
		0·3	
5	1·0		35
		−0·1	
6	1·1		41
		−0·2	
7	1·3		48
(b) For 50 bags of grain output			
1	—		—
		—	
2	4·2		52
		1·6	
3	2·6		41
		0·8	
4	1·8		38
		0·3	
5	1·5		40
		−0·2	
6	1·7		47
		−0·2	
7	1·9		54

price will remain unaltered. Other things being equal the price is likely to fall as the quantity offered for sale is increased. But a fall in the product price necessitates a fall in the economic optimum level of employment. Hence the ultimate effect of a change in productivity on the level of employment is dependent upon how easy it is to sell the extra product.

Two variable inputs and the least-cost combination

When more than one input can be varied we may distinguish two aspects of the farmer's production decision. One is how to combine the variable inputs, i.e. what method of production to use, and the other is how much to produce. We will deal with the first aspect now, assuming only two variable inputs, land and labour, and using the hypothetical relationships described in Chapter 1 and set out in Tables 1.3 and 1.4, pp. 12 and 17 and Fig. 1.4, p. 15.

With information on the rate of substitution between two resources and the respective price of each, the least-cost combination can be estimated. In Table 3.2 the total cost of each combination of land and labour is given in the final column, where labour costs £5 per unit and land costs £10. For 40 bags of grain the least total cost of £33 occurs where 3 or 4 units of labour and 1·3 or 1·8 units of land are used. Again, as in the case of a single variable input we may compare the extra cost of each additional unit of labour with the extra revenue it will earn, only in this case the extra revenue is represented by the saving in land cost. Thus the extra cost of each unit of labour is £5. The saving in land cost is more than this for the second and third units of labour at 4·2 × £10 = £42 and 1·0 × £10 = £10 respectively, but the saving in land cost for the fourth unit of labour 0·5 × £10 = £5 is exactly the same as the cost of the labour, so there is nothing to choose between 3 and 4 units of labour. The saving in land cost for the fifth unit of labour is only 0·3 × £10 = £3, which is less than the labour cost; therefore it does not pay to carry the substitution process this far.

At this 'least-cost combination of resources' the rate of technical substitution is equal to the inverse price ratio. The inverse price ratio, where price of labour is £5 and price of land is £10, is 0·5. The rate of substitution of labour for land is equal to 0·5 for the fourth unit of labour input. As we have seen this level represents the least-cost combination. For an output of 50 bags of grain the rate of technical substitution is 0·5 at about 4 units of labour and 1·8 units of land. This therefore represents the least-cost combination.

The argument may be reinforced by reference to the graph of the isoquants. In Fig. 3.2 straight equal-cost (or isocost) lines are superimposed on the graph. These lines represent all combinations of the two factors, land and labour, which could be purchased for a given total cost when the price of land is £10 per unit and the price of labour £5 per unit. For instance the line AB represents the isocost for £30 total outlay. This sum could be used to hire 3 units of land and no labour *or* 6 units of labour and no land, or any intermediate combination of land

and labour along the straight line joining these two points. Similarly the lines CDE and FGHIJ represent the isocosts for total outlays of £32 and £38 respectively.

The slope of the isocost is negative since, if total expenditure is kept constant, an increase in the quantity of labour hired must be accompanied by a decrease in the quantity of land. The negative of the slope therefore measures the number of units of land which can be hired for

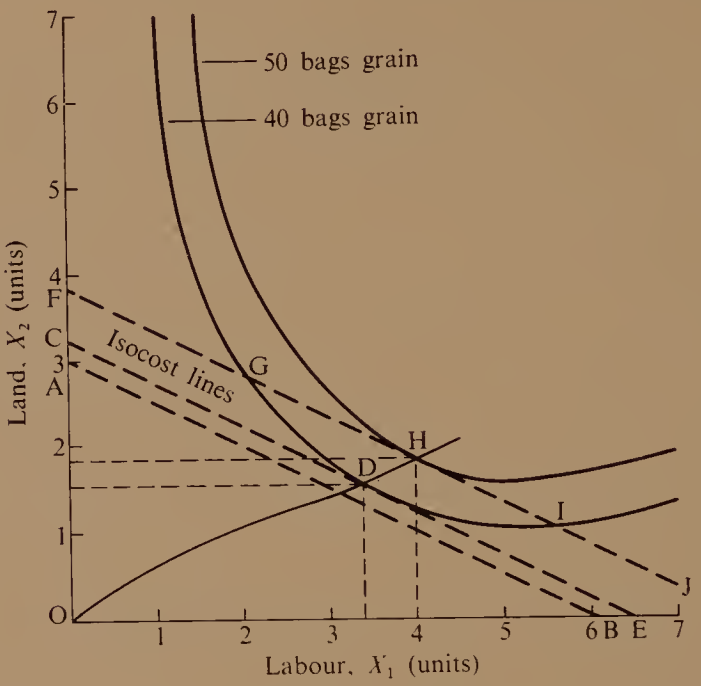


FIG. 3.2. The least-cost combination of two variable factors. This is found where the isocost line is tangent to the isoquant. Thus for 40 bags of grain output it consists of 3·4 units of labour and 1·55 units of land (point D). For 50 bags of grain output it consists of 4 units of labour and 1·8 units of land (point H). The expansion path (ODH) joins all least-cost points.

the price of one unit of labour, which depends upon the relative prices of land and labour. More precisely it equals the price per unit of labour divided by the price per unit of land. In the example this is £5 divided by £10, which equals 0·5. Thus the negative of the slope of the isocost line which measures the rate of value substitution is the inverse price ratio of the two resources. It is independent of the total expenditure. This means that if the relative prices of the two resources remain constant the ratio will remain constant regardless of the quantities used by the individual farmer. For this reason the isocosts are straight lines and parallel to each other.

It should now be clear that the least-cost combination of resources occurs where an isocost is tangent to (i.e. just touches) the isoquant. Thus the least-cost combination for an output of 40 bags of grain occurs at point D in the diagram, where total cost is £32. At any other combination of resources on this isoquant (e.g. point G or point I) the total cost is higher (e.g. £38 at G or I).

The least-cost combination for an output of 50 bags of grain occurs at point H, where the £38 isocost line just touches the isoquant. At this point the slope of the two lines is the same; the rate of technical substitution equals the inverse price ratio at the least-cost combination of resources. The graphical analysis therefore confirms our earlier conclusion.

For a given set of relative prices we can trace out the least-cost combinations for increasing levels of output. The line joining all least-cost combinations for a given set of prices is known as the 'expansion path'. A small part of it is represented by the line ODH in Fig. 3.2. It traces out the combinations of variable inputs a cost-minimizing farmer would use as he increases the output of his product.

It is also possible to define the least-cost combination in terms of marginal products. We saw in the first chapter that the rate of technical substitution of labour for land is equal to the marginal product of labour divided by the marginal product of land at any point on the production surface. Now, at the point of least-cost combination the rate of technical substitution of labour for land is also equal to the price ratio. It follows that the ratio of marginal physical products is equal to the price ratio at this point. This in turn means that the marginal product of labour divided by the price of labour is equal to the marginal product of land divided by the price of land; in other words the marginal product per £1 (or other unit of expenditure) is the same for each resource. This rule may now be applied to any number of variable inputs.

To illustrate: let us assume that the marginal product per £1 spent on labour or land is 2 bags of grain at the least-cost combination. Now if £1 of expenditure is transferred from land to labour, the marginal product of the extra £1 worth of labour will be less than 2 bags of grain, if the law of diminishing returns applies. But the output forgone by spending £1 less on land is 2 bags of grain. Thus any attempt to reduce costs by using less land and more labour, when the marginal product per £1 of cost is the same for each input, is likely to result in a fall in output and hence an increase in the cost per unit of output. The same is true if land is substituted for labour.

Naturally a change in the price of either resource would lead to a

change in the least-cost combination. If the price of labour were to increase in relation to the price of land, costs would be minimized by using less labour and more land to produce a given output. This may be checked by adding isocost lines to Fig. 3.2 for land at £10 per unit and labour at, say, £7.50 per unit. The converse would be true if the relative price of land rose.

New techniques of production may alter the shape of the isoquants so that a given level of output can be achieved with less of one or other or both of the resources. In the present context, and assuming constant prices, an innovation which proportionately reduces inputs of both labour and land in the least-cost combination, for a given level of output would be classified as neutral. One such as the introduction of irrigation, which tends to reduce the amount of land in the least-cost combination rather more than the amount of labour, would be called a land-saving innovation. Mechanization on the other hand is generally labour-saving.

The cost function and returns to scale

The analysis of the previous section defines the cheapest way of producing a given level of output; it does not determine the optimum level of output. In order to analyse this aspect of the farmer's decisions we now assume that output can be varied, but that each level of output is produced at minimum cost; the least-cost combination of resources is used. In other words, we assume that output is increased along the expansion path. Now, the minimum total cost can be determined for each level of output to give the cost function, which relates cost to output. From the total cost function we derive the average and marginal cost per unit of product.

Now, if *all* inputs can be varied it is possible to expand production to scale; i.e., to increase *all* inputs in the same proportion. Furthermore, if there are constant returns to scale, the expansion path must be a straight line through the origin. Once the least-cost combination of inputs has been found for one level of output, then any other level of output can be produced at minimum cost by changing all inputs in the same proportion. For instance, let us assume for simplicity that only two factors of production, land and labour, are required to produce grain.[5] Let us further assume that, as in the earlier example, the least-cost combination of labour and land needed to produce 40 bags of grain

[5] This is clearly an unrealistic simplification but the following discussion can readily be extended to cover more than two factors of production.

is 3·2 units of labour and 1·6 hectares of land. With labour priced at £5 per unit and land at £10 per hectare these inputs cost £32. Now, if there are constant returns to scale, doubling both inputs for example will double the output. This means that 80 bags of grain will be produced from 6·4 units of labour and 3·2 hectares of land, at a total cost of £64. Similarly 10 bags of grain will be produced from $3{\cdot}2/4 = 0{\cdot}8$ units of labour and $1{\cdot}6/4 = 0{\cdot}4$ hectares of land. The least-cost combinations for outputs ranging from 10 up to 80 bags of grain are set out in Table 3.3 and Fig. 3.3(a).

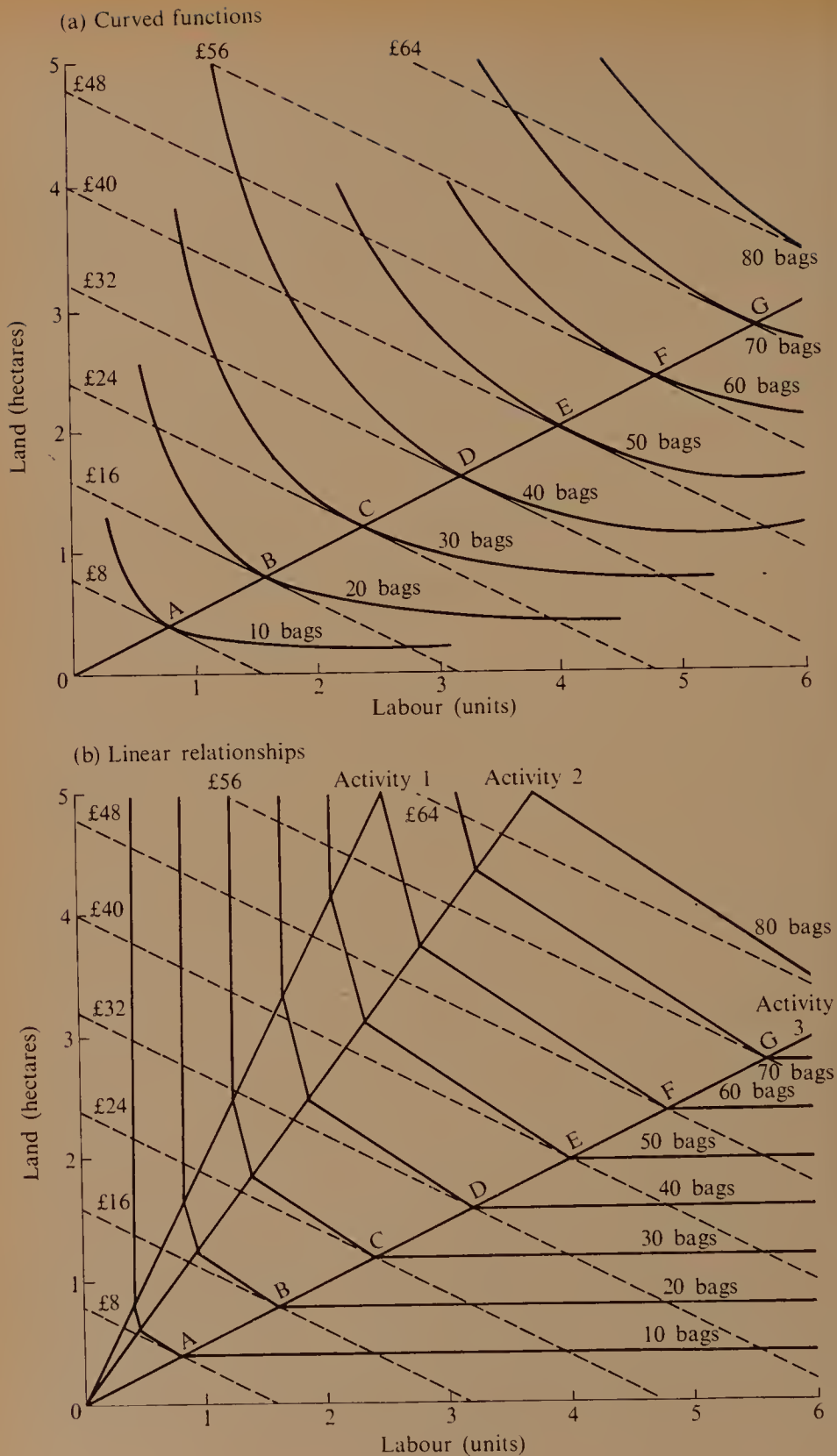
TABLE 3.3

The expansion path under constant returns to scale

Output of grain (bags)	Least-cost combination of inputs		Total cost: labour at £5 per unit and land at £10 per hectare (£)	Average cost per bag of grain (£)
	labour (units)	land hectares		
10	0·8	0·4	8	0.80
20	1·6	0·8	16	0.80
30	2·4	1·2	24	0.80
40	3·2	1·6	32	0.80
50	4·0	2·0	40	0.80
60	4·8	2·4	48	0.80
70	5·6	2·8	56	0·80
80	6·4	3·2	64	0.80

This result is now identical with that obtained when we assume straight-line production relationships. So long as there are no effective constraints the least-cost method of producing a particular product is likely to involve only one process, regardless of the level of output. Thus inputs are combined in the same proportions at all levels of output and the margin over cost per unit of output is constant. This is illustrated using the data from Table 2.1 (p. 41). If labour is priced at £5 per unit and land at £10 per hectare then activity 3 is clearly the cheapest method of producing grain at all levels of output, provided there are no effective constraints (Table 3.4).

The straight-line expansion path under constant returns to scale is illustrated in Fig. 3.3 both for curved functions and for linear production relationships, for the two-factor case. In both cases the least-cost combination, for each of the levels of output drawn on the graph, occurs where the isocost line just touches the isoquant at points A, B, C, D, E, F, and G. With the curved function, as we have seen, the isocost line is


(a) Curved functions
£56
£64
£48
£40
£32
£24
£16
£8
80 bags
70 bags
60 bags
50 bags
40 bags
30 bags
20 bags
10 bags
A
B
C
D
E
F
G
Land (hectares)
Labour (units)
0
1
2
3
4
5
6
(b) Linear relationships
£56
Activity 1
Activity 2
£64
£48
£40
£32
£24
£16
£8
80 bags
Activity 3
70 bags
60 bags
50 bags
40 bags
30 bags
20 bags
10 bags
A
B
C
D
E
F
G
Land (hectares)
Labour (units)

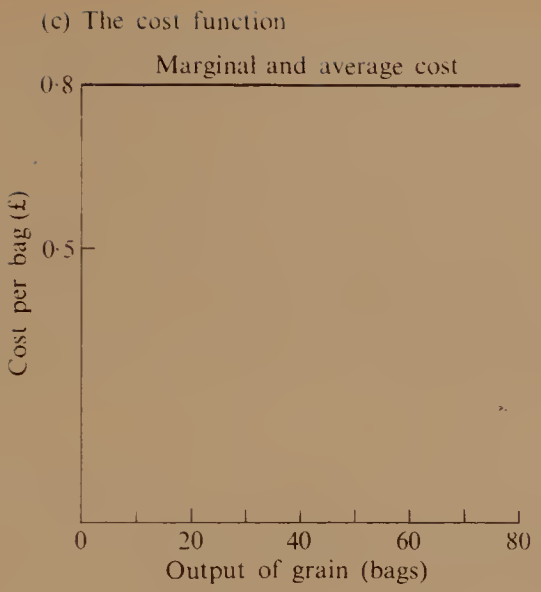


FIG. 3.3. Straight-line expansion paths under constant returns to scale. When inputs are increased to scale (i.e. in constant proportions) expansion may be represented by a ray through the origin. If there are constant returns isoquants are parallel and equally spaced along the ray representing the expansion path. The resultant marginal and average costs are constant.

tangent to the isoquant (at these points) so the two slopes are equal: the rate of technical substitution equals the inverse price ratio. This is not the case with the straight-line relationships of Fig. 3.3(b). Here the least-cost method is represented by a single activity, activity 3. Along the ray representing this activity the isocost line just touches the isoquant, but the rate of technical substitution is not defined so it cannot equal the inverse price ratio. This means that, whereas with curved isoquants even a slight change in the inverse price ratio will change the least-cost combination of resources, when there are straight-line relationships the least-cost combination is unaltered over a range of input price ratios. In this example, activity 3 is the cheapest method of production for labour: land price ratios from zero up to 0·57.

Clearly under constant returns to scale the average cost per unit of output is constant, in these examples at £0.80 per bag of grain. For this to be so, the marginal cost, which is the extra cost of producing an additional unit of output must also be constant and equal to the average cost. On a graph relating cost to output, both average and marginal costs are represented by a horizontal straight line as in Fig. 3.3(c). If grain can be sold for, say £1 per bag then the margin over cost is

£1 − £0.80 = £0.20 per bag. This margin can be earned on each bag of grain, no matter how many are produced. Thus there is no economic optimum, since profit can always be increased by increasing output.

The same would be true if there were increasing returns to scale. Then a proportionate increment in all inputs would cause output to expand more than proportionately so that the cost per unit of output (both average and marginal cost) would fall. Thus the margin earned on each bag of grain would rise as more were produced and again profit could be increased by increasing output. If, on the other hand, there were decreasing returns to scale, then marginal and average costs would rise as output grew and these rising costs would set a limit on profitable production.

TABLE 3.4

The least-cost activity (from Table 2.1)

	Activity 1	Activity 2	Activity 3
Output of grain (bags)	12	16	25
Input of labour (units)	0·5	0·75	2
Input of land (hectares)	1	1	1
Total cost: labour at £5 per unit and land at £10 per hectare (£)	12.50	13.75	20
Cost of grain per bag (£)	1.04	0.86	0.80

In practice it is unlikely that inputs can be increased to scale because some factors are indivisible while others are available only in limited quantities, thereby establishing constraints on productive activity. In either case expansion of output must then involve changing factor proportions; that is increasing the inputs of divisible, variable factors used in conjunction with each unit of the indivisible factor or the limited quantity of the constraining factor. Hence the expansion path will no longer be a straight line like those shown in Fig. 3.3(a) and (b), indeed it may not even be continuous. In such circumstances it is not appropriate to talk of returns to scale. However, any increase in output can be

described as an increase in size, whether factor proportions vary or not, so an increase in scale is a special case of an increase in size.

Factor indivisibility

Many agricultural resources are available only in whole units: they are chunky, or indivisible. For instance a tractor cannot be subdivided, neither can a combine harvester, a grain silo, or a man. Suppose 1 tractor can plough up to 50 hectares per year, then a farmer with only 10 hectares of arable land still requires the services of 1 tractor to plough his land. He cannot own $\frac{1}{5}$ of a tractor or plough unless he can arrange co-operative sharing with his neighbours. Similarly he cannot own a fraction of a combine harvester or of a grain silo, neither can he employ a fraction of a man. Of course the actual input of factor services, measured in tractor-hours or cubic metres of grain storage, for example, can be varied continuously and if these services could be hired by the hour or cubic metre as and when desired then costs would also be continuously variable. In many circumstances it is not possible to hire such inputs as and when desired and the farmer must employ, and meet the costs of, a whole number of tractors, combine harvesters, grain silos, and men.

In some cases indivisible resources are available in several different sizes; for instance there are small two- and three-wheeled tractors as well as four-wheeled tractors and crawlers. However, two- and three-wheeled tractors cannot do all the jobs that four-wheeled tractors do and invariably cost more to buy and operate per unit of power. Similarly, buildings such as grain silos can be constructed in various sizes but, up to a point, the larger they are, the cheaper they are per cubic metre of space. This is because the surface area of the walls does not increase as fast as the volume enclosed. For instance doubling the perimeter of a building increases the capacity four times.

Skilled labour is another form of indivisible resource. On a small farm the farmer must be a general worker competent at many tasks. On large farms with many employees there is more scope for specialization. The farmer or manager can concentrate and improve his skill at management and organization. Specialist cowmen, pigmen, and tractor-drivers may be employed. In this way each man is assigned to the task for which he is best fitted, where he becomes more expert through practice. Thus as the labour force grows there is more scope for increasing labour productivity through specialization and division of labour. Similarly, on a large farm, special-purpose equipment, such as planting and harvesting machines for root crops, can be justified more easily

than on a small farm. In short, because factors of production are indivisible the opportunities for specialization become more numerous as output increases.

Certain costs of an indivisible factor are fixed over a range of outputs up to its maximum capacity. Thus the purchase prices of a tractor and plough together with some annual costs such as tax and insurance are the same whether the farmer ploughs 10 or 50 hectares of arable land each year. Similarly many of the costs of owning a combine harvester are the same whether it is used to harvest 100 or 500 hectares. This means that the average cost per hectare or per unit of output must diminish as output is increased. To illustrate this let us assume that the annual fixed (indivisible) cost of owning a small combine harvester is £2 and it is capable of harvesting up to 40 bags of grain at a variable cost—made up of fuel and other operating costs—of 6p per bag.[6] When only 10 bags of grain are harvested the fixed cost is £2/10 = 20p per bag giving a total average cost of 26p per bag whereas when 40 bags are harvested the fixed cost is £2/40 = 5p per bag, giving a total average cost of 11p per bag. The average-cost curve derived in this way is shown in Fig. 3.4 by the solid line. On this basis it would appear that average cost per unit of output is minimized by operating at full capacity. Unused capacity results in high average costs.

The size of the enterprise, and hence total output, may be expanded by employing additional machines of the same kind. However, as the number is increased the effect of indivisibility on average costs becomes less significant. For instance if a second combine harvester, also of 40 bags capacity, is employed so that output may be raised above 40 bags then each machine will harvest at least 20 bags. The average fixed cost per bag will not exceed £2/20 = 10p so that with a variable cost of 6p per bag the over-all average cost will not exceed 16p. A single machine used to harvest less than 20 bags will incur a higher average cost. Once both machines are fully employed in harvesting a total of 80 bags of grain the average cost per bag is again 11p (see Fig. 3.4). In fact the operation of the second machine is just a duplicate of the operation of the first; it represents a simple doubling in scale.

As output grows, rather than replicating the original unit, it may be yet cheaper to use a larger and technically more efficient machine or to introduce more specialization and division of labour. In our example,

[6] These figures are clearly unrealistic. An annual fixed cost of £200 and a capacity of up to 4000 bags of grain might be feasible for a small machine when the cost per bag would be similar to the estimates given here. See Culpin, C. (1968). *Profitable farm mechanization.* Crosby Lockwood, London.

if the farmer wishes to produce 80 bags of grain it would be cheaper to use a large combine harvester with an annual fixed cost of £4 and a variable cost of 3p per bag than to use 2 small harvesters (see broken curve in Fig. 3.4). The cost per bag of grain when the large harvester is used would be £4/80 = 5p plus 3p giving a total of 8p as against 11p per bag for the smaller machines. Thus indivisibilities cause average costs to fall as output rises reflecting so called 'internal economies of scale'. However, it should be noted that the decreasing average costs are

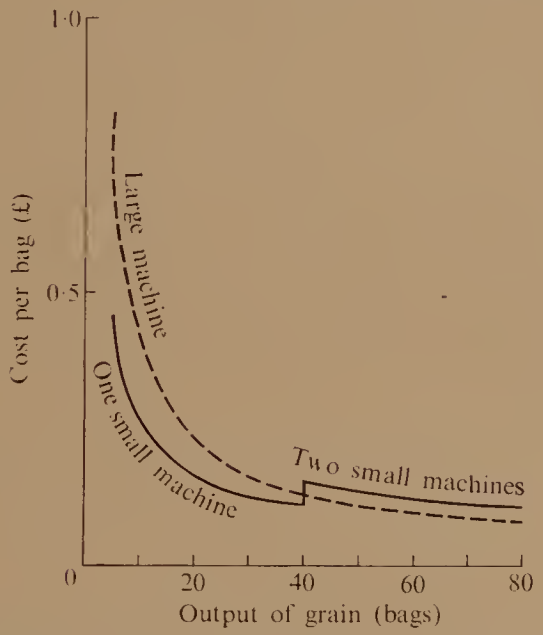


FIG. 3.4. Indivisible inputs and diminishing average cost. Given an indivisible input with an associated fixed cost (e.g. a machine) the average cost per unit of output must diminish as output is increased. The average cost of a larger machine (shown by the broken line) will be greater for small outputs but will eventually become smaller as output is increased.

the result not of increases in scale but of changing factor proportions as output is expanded.

Now, some would argue that there is no limit to the 'internal economies of scale' that may be reaped by expanding output, since there is always some unused capacity somewhere in the production system. When output is limited by combine harvester capacity there is surplus grain drying capacity; so an additional combine harvester is acquired which causes grain dryer capacity to become a limitation so that some combine harvester capacity is left unused. When combine harvesters and grain dryers are fully employed then labour is under-employed at certain

times of the year and so on. However there is no empirical evidence to suggest that average costs are any lower on very large farms than on those nearer the average size. We will assume that there is a 'minimum effective scale' of operation at which average cost reaches a minimum. Beyond this 'minimum effective scale' there are no 'internal economies' to be derived by further expansion.

Factor limitations

Generally, output and hence profit cannot be increased indefinitely without coming up against resource constraints or limitations. The limited resources are known as 'fixed factors' although, as we have seen, it may be perfectly feasible to vary the inputs of a 'fixed factor' up to the limit set by the amount available. There are obvious limitations on the total amounts of productive resources available to society as a whole over a given period of time. This is true not only of natural resources which cannot be replenished or increased, but also of labour, capital, and management within a given period of, say, one year. Indeed economics has been defined as 'the science which studies human behavior as a relationship between ends and scarce means' the latter referring to limited resources.[7]

Hence there is a problem for society as a whole of allocating these limited resources among competing uses and among individual producers. In some societies scarce resources are allocated according to the judgement of someone in authority—a tribal leader or an economic planner for instance; but in many parts of the world allocation takes place through an exchange or market system. Then, subject to prevailing prices, each producer is free to hire factors of production up to the limit set by the funds at his command. Prevailing market prices reflect the relative scarcity of factors of production.

Even where resources are allocated through the exchange system, there is a limit on the growth of the individual farm set by the rate at which funds can be raised. This in turn may depend upon the political power the individual can wield or upon assessments of his managerial and enterpreneurial ability. We will return to this topic in later chapters. For present purposes all we wish to establish is that there are, at any given point in time, constraints on the over-all size of an individual's productive activity. Furthermore there may be limits on the availability of land. Despite the existence of a market in land, it may prove difficult or impossible to obtain land in the desired location at the desired time.

[7] Robbins, L. (1932). *An essay on the nature and significance of economic science.* Macmillan, London.

Thus these factors set absolute limits on the growth of the farm business.

However, factors may be fixed in a relative sense depending upon the scale of the decision being taken. For instance, in order to decide upon the optimum usage (capacity) of a single combine harvester, the harvester itself represents a fixed or limiting factor but, if the problem is to decide how many combine harvesters should be employed, harvester inputs are then variable. For a major decision, such as whether to sell up the farm and move to another area, practically all factor inputs are variable. On the other hand, for a minor decision, such as when to weed the root crop, it is convenient to treat most factors as being fixed. Many economists draw the distinction in terms of time, major decisions being referred to as long-run decisions and minor ones as short-run decisions. But this terminology may be misleading. After all, a man could possibly sell his farm, move to another area, and start upon a new system of farming all within a few weeks.

Once a factor limitation becomes effective, further increases in output can be achieved only by changing the factor proportions; that is by using relatively more of those factor inputs that can still be varied. Then it is to be expected that the marginal (and average) products of the variable factors will eventually decline. But as the marginal product per unit of each variable input declines so the marginal cost per unit of that product must rise. In other words marginal cost varies inversely with marginal product (for proof see the mathematical appendix). Thus, once factor limitations become effective, marginal (and average) costs will eventually rise.

The total cost of a factor which has become an effective limitation on output is fixed; it does not increase with further increases in output. Hence marginal cost consists only of the cost of those inputs that can be varied. Other costs are fixed because of past decisions. For instance once land has been ploughed it is not possible to reverse the process and thereby recover the cost. Similarly once buildings are constructed or orchards established the cost becomes unavoidable and perhaps would be better referred to as an outlay. Contractual agreements such as those involved in hiring labour or land also cause resources to become committed before the productive process is started. Of course the outlays were made in the expectation that they would be recovered, but once they have been made they no longer affect current decisions. Only the variable costs; that is costs which vary as output is expanded, are relevant.

Sometimes a part of a fixed outlay may be recovered if things go wrong and the farmer wants to cut his losses. Ploughed land, buildings,

and orchards may be sold or hired out; contracts may be broken if compensation is paid. However, the salvage value is likely to be less than the original outlay, so only a part of the outlay is avoidable. We should note too that a factor which involves a fixed outlay may have various alternative uses within the farm; it hence incurs an opportunity cost. Leaving that possibility aside for the moment and considering a single enterprise, the economic optimum may be defined as the point where the surplus of total revenue over total variable cost is maximized. This surplus may be referred as the 'total gross margin', and represents the return to fixed factors plus the profit, if any.

Costs and the economic optimum

For illustration an imaginary cost schedule is set out in Table 3.5, the corresponding marginal and average cost curves being shown in Fig. 3.5. Note that we are now assuming that inputs of more than one factor may be varied and that they are increased along the expansion path.

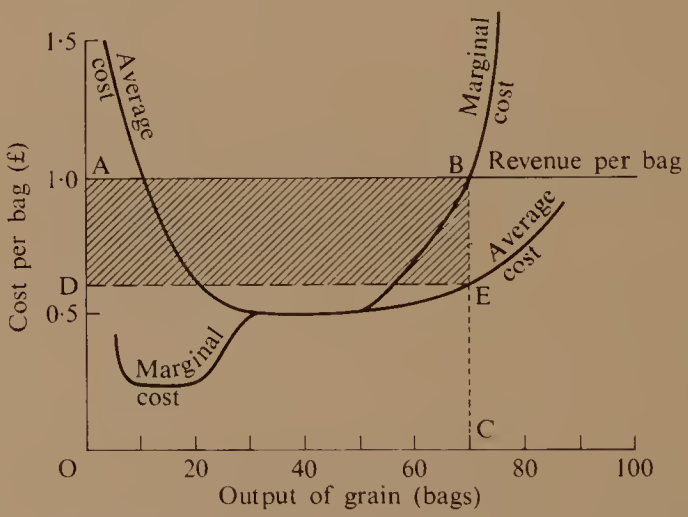


FIG. 3.5. Average and marginal cost curves. If there is any constraint or limitation on total output it is likely that marginal and average cost per unit of output will eventually rise, marginal cost rising more rapidly than average cost. Marginal cost is equal to average cost when the latter is at a minimum. Constant marginal revenue is shown by the horizontal line AB. Maximum profit is obtained where marginal cost equals marginal revenue at point B (OC units of output). Total gross margin is then given by area ABED.

Thus we assume that each level of output is produced as cheaply as possible; the least-cost combination of variable inputs is used. The cost curve is therefore a boundary or frontier separating inefficient cost

levels above the curve from infeasible ones below it. We could draw up such a cost schedule for a single variable input but it would add nothing to our understanding. We have already defined the economic optimum for a single variable input as the point where the value of the marginal product is equal to the unit factor cost.

TABLE 3.5
Cost function data

		Cost per bag of grain		
1	2	3	4	5
Output of grain (bags)	Total variable cost (£)	Average variable cost (£)	Marginal cost (£)	Total gross margin at product price of £1 per bag (£)
			1.00	
10	10	1.00		0
			0.20	
20	12	0.60		8
			0.30	
30	15	0.50		15
			0.50	
40	20	0.50		20
			0.50	
50	25	0.50		25
			0.70	
60	32	0.53		28
			0.80	
70	40	0.57		30
			1.60	
80	56	0.70		20

The average variable cost (column 3) is simply the total variable cost (column 2) divided by the output (column 1). The marginal cost (column 4) is the extra cost of producing one more bag of grain. It is obtained by dividing the differences between successive figures in column 2 (i.e. the increase in total variable cost) by 10 since in this table the costs are given at 10-bag intervals. Note, however, that the marginal cost, so defined, applies at a point somewhere within the interval. Thus the marginal cost of the tenth bag is neither £1.00 or £0.20 but probably somewhere in between. The marginal cost of every individual bag between 5 and 75 is given in Fig. 3.5.

The relationship between marginal and average costs is the mirror image of that between marginal and average products. As long as marginal cost is less than average cost, average cost must fall as output is increased. When marginal cost exceeds average cost, then the latter must rise as output is increased. Marginal and average costs are equal when average cost is at a minimum. For average costs to remain constant

at this minimum level marginal cost must remain equal to average cost as output is expanded.

The data given in Table 3.5 and graphed in Fig. 3.5 are designed to show three stages. The first stage up to 30 bags of grain output is one of declining average costs as a result of factor indivisibilities. Note that we are now treating the costs of the indivisible items as variable costs. An output of 30 bags of grain is therefore the 'minimum efficient scale.' In the second stage between 30 and 50 bags of grain output, inputs are increased to scale and there are constant returns, reflected in constant and equal marginal and average costs. Finally the third stage shows increasing marginal costs as a result of factor limitations. As marginal cost rises above average cost then the average cost must itself rise. It is in this stage of increasing marginal costs that the economic optimum rate of output is found.

The economic optimum occurs at the point where the marginal cost of increasing output is just equal to the marginal revenue, which is the extra revenue gained by producing and selling one more unit of output. Since we are assuming that the price per unit of product output is constant, marginal revenue is equal to the product price. For the costs of production given in Table 3.5, and for a product price of £1 per bag, the economic optimum occurs at an output of 70 bags of grain where marginal cost is also £1 per bag. At this rate of output the surplus over variable costs, or gross margin, per unit of output is £1.00–£0.57 = £0.43. Thus the total gross margin is £0.43 × 70 = £30. Note that at the economic optimum, average cost is not minimized so gross margin per unit of output is not maximized. The average variable cost is only £0.50 between 30 and 50 bags of grain output so the average gross margin is also £0.50, but the total gross margin when 50 bags of grain are produced is only £0.50 × 50 = £25.

The economic optimum rate of output is also shown in Fig. 3.5 where the horizontal line AB represents the marginal revenue or price per bag of £1. This line crosses the rising portion of the marginal cost curve, that is marginal cost is equal to marginal revenue, at point B representing 70 bags of grain output. The total revenue of £70 is represented by the area ABCO. The total variable cost equals the average variable cost per bag times the number of bags produced so it is measured by the area DECO, which represents £0.57 × 70 = £40. The total gross margin which is the difference, £70–£40 = £30, is given by the shaded area ABED. Since the fixed cost is assumed not to vary with output the point of maximum gross margin is also the point of maximum profit.

Remember the average variable cost for any level of output is deter-

mined at the least-cost combination of resources, which means that the marginal product of the last £1 spent is equal for all variable factors. To arrive at the economic optimum rate of output these conditions must be met simultaneously with the condition that marginal cost equals marginal revenue or product price. It then follows that for each and every variable factor the value of the marginal product will be equal to the unit factor cost, (for proof see mathematical appendix to this chapter). Thus our rule for the economic optimum rate of use of a

TABLE 3.6

Output and costs of alternative activities

	Activity 1		Activity 2		Activity 3
Grain output per hectare (bags of grain)	12		16		25
Output from 3 hectares (bags of grain)	36		48		75
Extra output of grain (bags)		12		27	
Labour cost per hectare (£)	2.50		3.75		10
Labour cost on 3 hectares (£)	7.50		11.25		30
Extra labour cost (£)		3.75		18.75	
Average variable cost per bag of grain (£)	0.21		0.23		0.40
Marginal cost per bag of grain (£)	0.21	0.31		0.69	

single variable input can be applied more generally to situations where more than one input is varied.

The economic optimum can only be defined as the point where marginal cost equals marginal revenue (or nearly so), if marginal cost is rising. Then, as output increases towards the optimum, marginal revenue is greater than marginal cost but beyond this point marginal cost exceeds marginal revenue. Clearly, if the marginal cost curve was falling, then profits would be increased or losses would be reduced by increasing output beyond the point where marginal cost equals marginal revenue. The extra cost of doing so would be less than the extra revenue.

Rising marginal costs may occur even in the case of linear production relationships if there are resource constraints. This may be illustrated using the data from Table 2.1 (p. 41) and assuming that labour inputs are variable at a price of £5 per unit but that land resources are fixed at 3 units. The maximum output which can be achieved with each activity and the corresponding variable costs are given in Table 3.6. For a single activity marginal cost is equal to average variable cost but as one activity is substituted for another marginal cost will differ from average

variable cost. Thus up to 36 bags of grain may be produced using activity 1 alone at a constant marginal and average variable cost of £7.50/36 = £0.21. The land constraint then becomes effective so that further increases in output can only be achieved by switching to activity 2 which is more costly in terms of labour. Activity 2 alone will produce 48 bags of grain from 3 hectares of land at a total variable cost of £11.25, which represents an average variable cost of £11.25/48 = £0.23 per bag. As activity 2 is substituted for activity 1 average variable cost rises from

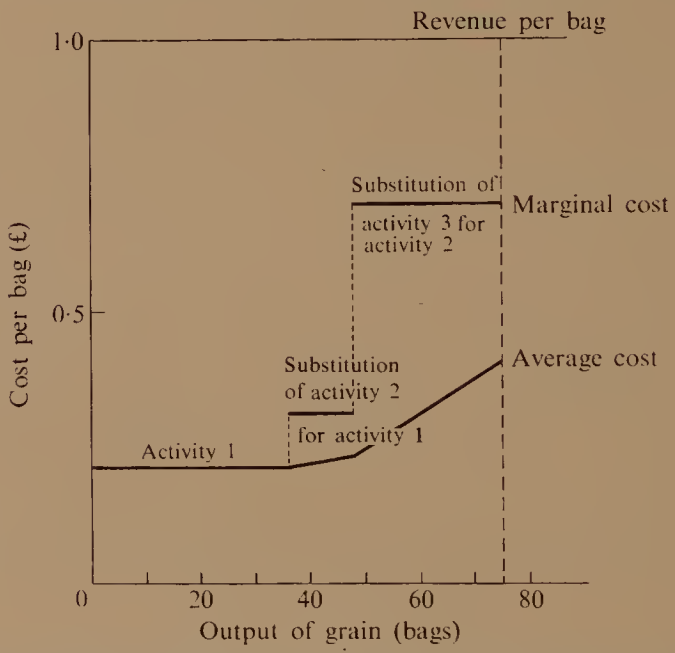


FIG. 3.6. Increasing marginal cost with linear relationships. In this case marginal cost rises in discrete steps as one activity is substituted for another.

£0.21 to £0.23. But the extra cost of 48 − 36 = 12 bags of grain is £11.25 − £7.50 = £3.75 so the marginal cost is £3.75/12 = £0.31. Similarly, although the average variable cost per bag produced by activity 3 alone is £30/75 = £0.40, the marginal cost as output is increased from 48 to 75 bags is £18.75/27 = £0.69. The resultant stepwise increasing cost function is shown in Fig. 3.6.

Given just these three alternative activities and given a product price of £1 per bag, the economic optimum output is 75 bags produced by activity 3. However, because of the discontinuities, the condition of equality between marginal cost and marginal revenue is not met. Indeed at the economic optimum the marginal cost is not defined. Such an optimum solution is known as a 'corner' or 'boundary optimum'.

In fact corner optima may occur even when the marginal cost curve rises smoothly. For instance, if the product price is so low that it does not pay to produce any output we have a corner optimum at zero output. Alternatively, for a product which is profitable to produce, there may be certain factors of production for which the optimum input is zero. This is likely to be the case when there are perfect substitutes (see Fig. 1.6, p. 19). In such cases the 'least-cost combination' consists only of the cheaper factor. If we assume that tractors are perfect substitutes for horses, and if tractor power is cheaper, then the optimum input of horses is zero.

Where there are indivisible inputs that can be varied only in large steps exact equality of marginal cost and marginal revenue may not be possible. Similarly exact equality of the marginal product per unit of expenditure on each factor of production may not be possible. Nevertheless in order to maximize profits the number of indivisible units should be so adjusted that:

(*i*) the loss of revenue resulting from the use of one less unit would be greater than the saving in costs;
(*ii*) the extra revenue resulting from the use of one more unit would be less than the extra costs.

In other words the level of input should be such that by no possible readjustment is it possible to increase profits.

Product supply and factor demand

Now if we assume that producers always maximize profits we may use a cost schedule such as that presented in Table 3.5 and Fig. 3.5 to estimate the relationship between product price and quantity produced, which is known as the individual supply function. The aggregate supply function of a given product is the horizontal sum of all the individual supply functions for that product. Since we assume that the individual farmer adjusts his output so that marginal cost equals the product price, the relationship between output and product price (the supply curve) must be the same as the relationship between output and marginal cost (the marginal cost curve) where the latter is rising. Instead of using the schedule or curve to read off the marginal cost for a given level of output we can use it the other way round to read off the output for a given price level. For example, from the last 3 rows of Table 3.5, it would appear that a profit-maximizing farmer would produce 60 bags of grain at a price of £0.75 per bag, 70 bags of grain at a price of £1.00 per bag, and 80 bags of grain at a price of £1.60 per bag. However,

if the product price should fall below the minimum average variable cost the farmer would minimize his losses by discontinuing production, since the revenue he would forgo by so doing is less then the variable cost he would avoid.

In our example, if the price of grain falls below £0.50 it is more profitable to stop producing grain even if certain fixed costs still have to be paid. Thus the quantity of product supplied is zero for prices lower than the minimum average variable cost and for prices above this the supply curve is identical with the marginal cost curve. It should be noted that this assumes there are no alternative uses and hence no opportunity costs for the fixed factors. If the fixed factors could earn some revenue other than in grain production, then it would be profitable to abandon grain production before the price fell to the level of minimum average variable cost. In fact total revenue would have to cover variable cost and the opportunity cost of the fixed factors.

We have so far assumed that the individual farmer's costs of production are dependent only on his own rate of output. In fact his costs may well depend upon the combined output of all producers as a result of so-called 'external scale effects'. These effects may be pecuniary, that is price effects, or technological. For instance, suppose a rise in grain price stimulates an expansion in grain production. Now as more and more land, labour, and other resources are drawn into grain production, their opportunity costs and indeed their market prices may rise (pecuniary effects). At the same time the general expansion of grain production may encourage the spread of cereal pests and diseases so that more must be spent on pest control (a technological effect). As a result the whole cost curve for each individual producer is raised. In such a case there are 'external diseconomies of scale'; grain production would be a 'rising cost industry'. The effect on the individual and aggregate supply functions will be to reduce the rate at which output is expanded in response to rises in product price. Indeed if marginal cost should rise by the same proportion as product price does, there would be no change in the optimum level of output for a single producer.

It is also possible for there to be 'external economies of scale' which would cause the individual producer's cost curves to shift downwards as aggregate output is expanded. External economies arise, for instance, if the suppliers of inputs such as fertilizers, machinery, or animal feeds operate under increasing returns to scale. Then, as the quantities of inputs they supply are increased, they are able to reduce their prices. Thus farmers may benefit from pecuniary external economies of scale. Technological external economies result, in the main, from the greater

opportunities for specialization and division of labour between individual producers as aggregate output is increased. However, it seems most likely that external diseconomies will eventually outweigh external economies so that the individual producer's cost curves will eventually rise.

The individual derived demand function for each variable input, which relates the quantity of the input employed to the prices of the product and all variable factors, can be estimated from the production function in a similar manner. Our analysis suggests that if product price rises relative to all factor prices the inputs of variable factors will increase. The effect is the same if factor prices fall relative to product prices. However, if factor prices vary in relation to one another the effects are less predictable. Thus if the price of fertilizer rises in relation to the price of seed, the least-cost combination of these two factors will shift in the direction of using less fertilizer and more seed. This is known as the 'substitution effect'. At the same time the increase in fertilizer price will result in an upward shift in the variable cost curve, so that if the product price stays constant the optimum level of inputs and output will be reduced. This is known as the 'output effect'. The combined effect is clear in the case of the fertilizer, since the substitution effect and the output effect both operate in the same direction to reduce optimum input as the price rises.

However, the substitution and output effects act in opposite directions on the optimum input of seed. As fertilizer price rises the substitution effect will tend to cause an increase in the input of seed, whereas the output effect will tend to reduce its use. The direction of change can be predicted if the form of the production function is known, but it must depend upon the extent to which one factor can be substituted for the other. In this case it is doubtful whether seed can be substituted for fertilizer in maintaining a given level of output, so the output effect is likely to be stronger and seed-use is likely to fall with increasing fertilizer price. The economic optimum level of use of each variable input is still found where the value of the marginal product equals the unit factor cost, but, because the marginal product per unit of one factor depends upon the level of input of another factor, the economic optimum for one factor depends upon the prices and hence levels of use of other factors.

The return to limited factors

Our assumptions regarding the supply of limited fixed factors are quite different from those regarding the supply of variable factors. For

the individual producer we have assumed that the price of a variable factor is not affected by the amount he uses, as shown in Fig. 3.7(a). He can obtain all that he wants at the going price, OP, but if he offers less than this price he will not get any. Inputs of a fixed factor, on the other hand, are subject to a maximum limit OM in Fig. 3.7(b), which is not dependent upon the price the farmer is willing to pay. The broken horizontal line at OR encloses an area ORQM, representing the cost of acquiring OM units of the services of the fixed factor but, once the limitation is effective, the rate of return to the fixed factor may be totally independent of its cost of acquisition.

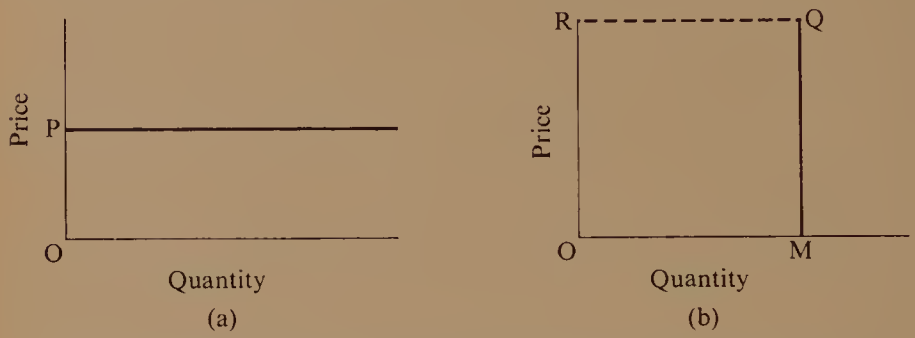


FIG. 3.7. The supply of variable and fixed factors to the individual farmer. (a) For variable factors there is assumed to be no limit on the amount the individual can aquire and price is independent of the amount he uses. (b) For fixed factors there is an absolute limit on the services available regardless of price. Cost of aquisition ORQM is irrelevant to future decisions.

Thus it could reasonably be argued that the whole surplus over variable costs, i.e. the total gross margin, represents the return to fixed factors, or the amount the fixed factors earn. For example, from Table 3.5, we see that when 70 bags of grain are produced the total variable cost is £40 and, since the total revenue is £70, the gross margin is £30. Now, if this productive activity were abandoned completely, the variable cost would be avoided, and £40 worth of variable inputs would be available for other uses each year. The total annual revenue of £70 would be forgone, so the opportunity cost of releasing the fixed factors would be £30, which is the total gross margin. Thus we are justified in ascribing the total gross margin to the fixed factors regardless of their cost of acquisition. If we think of the return to fixed factors as being equivalent to the total gross margin, it clearly depends upon the level of variable inputs and the market prices of the product and variable inputs. For a given set of prices the return is maximized at the economic optimum level of output. If product price should rise relatively, or

variable input prices should fall, the return to fixed factors would rise for two reasons; first because the total gross margin would be increased for a given level of output and secondly because the optimum level of output would also increase.

The marginal rate of return to individual fixed factors may be imputed provided that technical substitution is possible. Then for a given fixed factor we can define, and perhaps estimate, the marginal product of a one unit increase in the quantity available, that is the extra product obtained by relaxing the constraint by one unit. The value of this marginal product then represents the maximum price which the farmer could afford to pay in order to acquire an extra unit of the fixed factor.

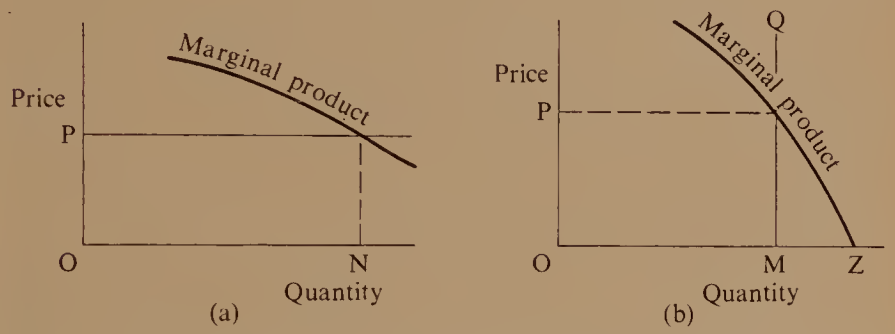


FIG. 3.8. The opportunity costs of (a) variable and (b) fixed factors. For variable factors the opportunity cost is the market price which is determined externally; equilibrium level of output is dependant upon price. For fixed factors the opportunity cost is the shadow price which is determined internally; price is dependent upon equilibrium level of output.

It is known as the 'shadow price'. Alternatively the shadow price may be thought of as an estimate of the marginal product of the last available unit of the fixed factor. It then represents the opportunity cost of transferring that unit to some alternative use.

For an individual farmer, the price of a variable factor is determined externally and he can make the value of the marginal product equal to the factor price only by adjusting the rate of use as shown in Fig. 3.8(a). But the shadow price of a fixed factor is determined internally and must be equal to the value of the marginal product when the constraint is effective (Fig. 3.8(b)). For some limited factors the value of the marginal product may fall to zero before the limit is reached; that is before the constraint becomes effective. It would be irrational to increase inputs beyond the point of zero marginal product with the result that some units of such factors are left unused, or in disposal. In such cases the shadow price is zero. This could be illustrated in Fig. 3.8(b) by moving the

vertical line QM to the right of point Z. Note that all these conclusions regarding shadow pricing were originally developed in the linear programming context but they apply equally when there are curved (non-linear) production relationships subject to constraints.

Now the total return to a given fixed factor, say land, may be imputed by evaluating each unit available at the shadow price. However it may not be possible to estimate the return to all fixed factors in this way because some may be indivisible as well as being limited. In particular this is true of management. It might appear that the return to such a factor is the difference between the total gross margin when the factor is employed and the total gross margin when it is not. But this does not allow for the interaction between inputs. The total output and hence total gross margin in the absence of management is likely to be zero. Thus the whole of the gross margin would then be imputed to management. If other constraints are effective at the optimum rate of output, this assumption must be false. Instead the surplus, or deficit, remaining, when other factors are ascribed returns based on their shadow prices, may be viewed as the return to management.

The total return to fixed factors, i.e. the total gross margin, is made up of two parts. On the one hand there is the amount that each factor must earn in its present use in order to prevent it from transferring to another use. This is sometimes called the factor's 'transfer earnings', though clearly it is the same as the factor's opportunity cost in its present use. The transfer earnings of management are sometimes referred to as 'normal profit' but they are more properly viewed as costs of production. On the other hand there may be a surplus over and above these transfer earnings which is pure profit.[8] How much of the return to a given factor is transfer earnings and how much is pure profit depends on the time period under consideration and the extent to which transfers are feasible.

It is often argued that, under competitive conditions, pure profits will be eroded away. This is because the existence of profits will attract new individuals into the industry. For instance, suppose there are pure profits to be gained in grain production and in consequence more farmers start growing grain. This means that the total output of grain will rise and so too will the quantities of variable inputs used in grain production. Eventually external dis-economies of scale will cause the market price of grain to fall and the variable costs of grain production to rise. These processes, it is argued, will continue until there are no

[8] This surplus is frequently called 'economic rent' but we prefer to avoid this terminology since in common usage 'rent' means the opportunity cost or transfer earnings of land.

pure profits to be earned in grain production. By the same token, if profits should fall below zero, that is if losses are made, farmers will abandon grain production and output will fall until the point of zero pure profit is again reached. At this point, which is often called the long-run equilibrium, the shadow price of each fixed factor is equal to its transfer earnings.

However, in practice, it is unlikely that all pure profits (and losses) will disappear. Many farmers would be unwilling to face the insecurity associated with being self-employed if they could earn just as much by hiring out their resources to someone else: that is if they were not earning pure profits. Others derive so much satisfaction from farming that they are prepared to accept a lower income than they could expect to earn elsewhere; they accept continuous losses. Furthermore farmers vary in their willingness and ability to take risks and innovate. Those who first adopt a more efficient technique of production will earn profits. Given time the new technique will spread to other farmers and, in the manner discussed above, the profit to be gained from that innovation will be eroded way. (More precisely the gains from the greater efficiency will have been transformed into a fall in the product price and an increase in factor earnings.) However, the progressive farmer, by repeated innovation, may continue to earn more profit than his neighbours. In short the factor 'entrepreneurship' may command a price although it involves no transfer earnings since it cannot be transferred from one individual to another.

We will return to this topic in Chapter 8. We shall simply point out here that the theoretical zero profit equilibrium is unlikely ever to be attained in practice.

Product exhaustion under constant returns to scale

It is instructive to consider the special case where there are no indivisibilities and all inputs can be varied. The return to each and every factor may then be defined in terms of its marginal product. The question which then arises is: if each factor is rewarded according to its marginal product, will there be any surplus or deficit or will the total product be exhausted exactly? The answer is that where there are constant returns to scale the product will be exhausted exactly. Where there are decreasing returns to scale a surplus will occur and where there are increasing returns a deficit will occur.

We saw this in physical terms in the previous chapter, where a combination of 0·8 hectares of an activity yielding 16 bags of grain per hectare, and 2·2 hectares of a second activity yielding 25 bags of grain

per hectare, exactly account for 3 hectares of land, with a marginal product of 10·6 bags of grain per hectare and 5 units of labour with a marginal product of 7·2 bags of grain per unit. The total product is

$$(0{\cdot}8 \times 16) + (2{\cdot}2 \times 25) = 67{\cdot}8 \text{ bags of grain,}$$

which can be imputed to the two factors land and labour in accordance with their marginal products:

$$(3 \times 10{\cdot}6) + (5 \times 7{\cdot}2) = 67{\cdot}8 \text{ bags of grain.}$$

This relationship still holds if we evaluate the grain in money terms. For simplicity we may evaluate it at £1 per bag. We then find that total value is:

$$(0{\cdot}8 \times £16) + (2{\cdot}2 \times £25) = £67{\cdot}8,$$

which may be imputed as returns to 3 units of land and 5 units of labour as follows:

$$(3 \times £10{\cdot}6) + (5 \times £7{\cdot}2) = £67{\cdot}8.$$

Thus the total supply of land, evaluated at its marginal product and the total supply of labour evaluated at its marginal product, together exactly account for the total revenue. This result holds for any productive activity showing constant returns to scale.

The reader may have noted that in this simple example land and labour are limited factors or constraints. Strictly speaking the return of £1 per bag of grain is the gross margin. The calculations actually demonstrate that when limited factors are evaluated at their shadow prices and there are constant returns to scale, the total gross margin will be exhausted exactly. However if we assume that variable (unconstrained) factors are used at the economic optimum rate it follows that they are rewarded according to the value of their marginal products. Then if the gross margin which remains is exactly exhausted so too must be the total revenue product. This result, that the total product is equal to the sum of the amounts of the factors each multiplied by its marginal product, applies to any productive enterprise which is subject to constant returns to scale.

However, this result must not be taken to imply that pure profit can never be earned under constant returns to scale. The value of the marginal product of a fixed factor may well exceed the opportunity cost or cost of acquisition. Thus if the value of the marginal product of land is estimated at £10.6 per hectare, that of labour at £7.2 per unit, and if land costs only £10 per hectare and labour only £5 per unit to hire then clearly profits are made: £0.6 on every hectare of land in use and £2.2

on every unit of labour employed. This result suggests that the farmer should try to employ more land and labour, as long as the value of the marginal product exceeds the unit factor cost. However, we have assumed that this is not possible in the case of fixed factors; indeed, this is why they are distinguished from variable factors. If they were variable, of course, the farmer could in theory go on expanding output, and hence profit, indefinitely, as long as market prices and opportunity costs remained constant.

Alternative products and opportunity costs

The optimum allocation of a limited factor between two alternative enterprises is found where the value of the marginal product is the same in both enterprises. This may be illustrated with the production-response data for labour in grain and root production given in Table 1.5 (p. 22). If grain is priced at £1 per bag and roots at £3.50 per tonne the values of the marginal products will be as set out in Table 3.7. At a

TABLE 3.7

Values of marginal products in two alternative enterprises

Labour (units)	Value of the marginal product in grain at £1 per bag (£)	Value of the marginal product in roots at £3.50 per tonne (£)
1	7	21
2	13	21
3	11	21
4	6	21
5	3	17.50
6	−1	3.50
7	−4	nil
Total revenue from 7 units of labour	35	105
Average value product	5	15

wage rate of £5 per unit, the economic optimum level of labour use in grain production would be about 4 units. However, since we have assumed that only 7 units of labour are available in total, the use of 4 units in grain production would leave only 3 units for use in root production. Now, labour-use in root production is below the economic optimum, which occurs at about 5 units of labour input. Hence the two enterprises compete for the scarce labour resources and labour has an opportunity cost within the farm, which differs from the wage rate.

It is clear from Table 3.7 that if the enterprises were mutually exclusive, total revenue would be maximized by devoting all 7 units of labour to root-crop production. However, if a combination of both enterprises is possible, total revenue can be increased by transferring labour into grain production. The first unit of labour used in grain production earns £7 and the opportunity cost of removing the seventh unit of labour from the root-crop enterprise is zero. The second unit of labour transferred into grain production earns £13 at an opportunity cost of £3.50. However, if a third unit of labour is transferred the additional revenue is £11 while the opportunity cost is £17.50. Thus total revenue is increased by transferring the first two units of labour into grain production but not the third. This leaves 5 units of labour in root production and 2 units in grain production. The total revenue is then £121.50.

The same conclusion is reached if we consider a unit by unit increase in labour use. To maximize profits, each unit of labour should be used where it will add most to revenue. Therefore the first unit should be used in root-crop production where it will earn £21. Similarly the second, third, fourth, and fifth should all be used on the root crop. However, the sixth unit would earn only £3.50 in root production whereas if used in grain production it will earn £7. Thus this unit of labour should be used in grain production. The seventh unit of labour should also be used in grain production where it will earn £13 as opposed to the £3.50 it would earn if used to produce roots.

If the labour inputs in each enterprise could be varied continuously then the most profitable allocation would occur where the value of the marginal product is equal in each enterprise. Since the steps by which labour is varied are quite large in our example, the values of the marginal products are not equal, but they are as nearly as possible. This value of the marginal product which is the same in all uses then represents the shadow-price of labour. It is greater than the market wage rate or transfer earnings of £5 per unit. The principle involved in allocating resources between alternative uses is therefore known as the equal marginal returns principle.

The same result is obtained if we use the production-possibility curve, tabulated in Table 1.6 (p. 23) and drawn in Fig. 1.7 (p. 24). In Table 3.8 the total revenue from each combination of roots and grain is given in the last column, with grain priced at £1 per bag and roots at £3.50 per tonne. As we have already seen, the maximum total revenue of £121.50 occurs where 5 units of labour are used in root production and 2 units are used in grain production, i.e. where 29 tonnes of roots and 20 bags

of grain are produced. Costs are unaffected since 7 units of labour are employed for all the alternative combinations, so total profit is maximized at the point where total revenue is maximized.

This maximum profit point may be defined in terms of the rate of product transformation in a manner which is similar to the definition of the least-cost combination of resources. Thus the maximum profit

TABLE 3.8

Choice of products

Price of grain, £1 per bag; price of roots, £3.50 per tonne; hence inverse price ratio is 3·50/1 = 3·5

Total physical product, roots (tonnes)	Total physical product, grain (bags)	Rate of product transformation: bags of grain per tonne of roots	Total revenue (£)
0	35		35
		−0·7	
6	39		60
		−0·2	
12	40		82
		0·5	
18	37		100
		1·0	
24	31		115
		2·2	
29	20		121.50
		13·0	
30	7		112
		∞	
30	0		105

combination of enterprises occurs where the rate of product transformation of grain for roots equals the inverse price ratio. In the present example the inverse price ratio is 3·50/1·00 = 3·5. The rate of product transformation is nearest to this value where 5 units of labour are used on the root crop and 2 units are used in grain production. As we have seen, this is the most profitable allocation of labour between these enterprises.

Finally, this profit-maximizing position is illustrated graphically in Fig. 3.9, which represents the production-possibility curve from Fig. 1.7 together with iso-revenue lines. These lines join all combinations of roots and grain which yield the same total revenue. The lowest iso-revenue line shown in the diagram represents all combinations of roots and grain which yield a total revenue of £35. This can be obtained from

£35/£3.50 = 10 tonnes of roots or £35/£1 = 35 bags of grain or any combination of the two products along this line. The next iso-revenue line shown in the diagram is the £70 iso-revenue line. The third one is for a total of £121.50. All iso-revenue lines are parallel provided that the relative prices of the products do not change.

The slope of the iso-revenue lines is negative because for total revenue to remain constant, increased revenue from one product must be associated with decreased revenue from the other. In fact the number of bags of grain which must be sold to earn the same revenue as a tonne of roots de-

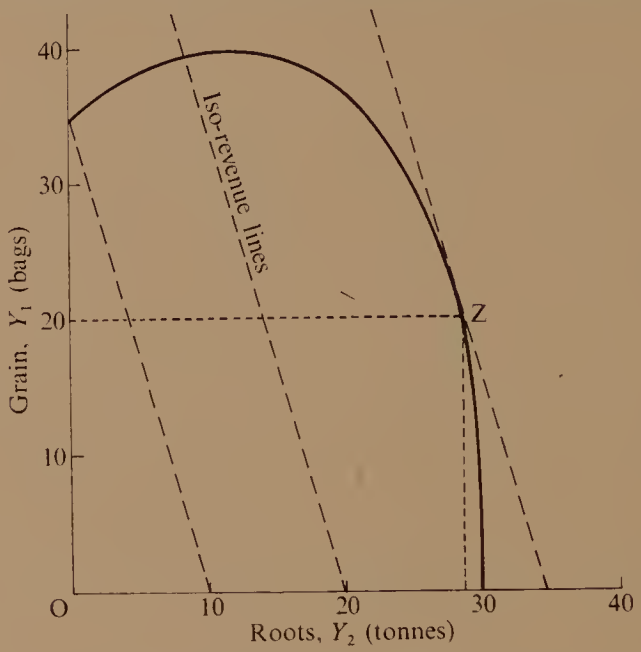


FIG. 3.9. The economic optimum product combination. The maximum possible revenue is obtained at point Z where the iso-revenue line is tangent to the production-possibility curve.

pends upon the relative prices of grain and roots. More precisely, it equals the price per tonne of roots divided by the price per bag of grain. In the example this is £3.50 divided by £1 which equals 3·5. Thus the slope of the iso-revenue line is the negative inverse price ratio of the two products.

The maximum revenue that can possibly be obtained is £121.50 since the iso-revenue line for this sum of money just touches (is tangent to) the production-possibility boundary. This is the highest 'rung' that can be attained on the 'step ladder' of increasing revenue. All other feasible product combinations on, or within, the boundary must lie on lower-valued iso-revenue lines. Thus the optimum product combination

is found at point Z where 20 bags of grain and 29 tonnes of roots are produced.

The maximum revenue combination of enterprises would be altered by any change in the price ratio of the two products. If the relative price of roots fell it would pay to produce less roots and more grain. It should be noted, however, that for the production-possibility curve given, it will never be profitable to concentrate entirely on one enterprise, i.e. to specialize. Even if the price per tonne of roots rose to say 100 times the price per bag of grain it would still pay to produce at least 7 bags of grain. Conversely if grain price rose to a very high level in relation to the price of roots it would still pay to produce 10 tonnes of roots. This may be checked by adding iso-revenue lines to Fig. 3.9 for various alternative price ratios. This advantage in favour of diversification, or producing both enterprises, is due to supplementary and complementary relationships between them. As it was pointed out in Chapter 1, the area of rational choice lies between the ridge lines where the slope of the boundary is negative.

Any technical change which altered labour productivity in one enterprise more than in the other would alter the optimum combination of products. For instance, if labour productivity in grain production increased more than labour productivity in root production, proportionately more grain would be included in the optimum product combination.

The linear programming solution

In linear programming we are concerned with situations where enterprises compete for limited factors but where the input–output relationships are assumed to be linear. Where there are only two alternative products (activities) a graphical method of solution may be used that is practically identical with that for curved functions. Fig. 3.10 shows the production-possibility boundary already illustrated in Fig. 2.12 (p. 57) but with the iso-revenue lines added, for prices of £1 per bag of grain and £3.50 per tonne of roots. In this case it is clear that the highest possible revenue of £126 is earned at point M where 21 bags of grain and 30 tonnes of roots are produced. From Fig. 2.12 we find that at this point only the total labour and the December labour constraints are effective. Some June labour and some land are in disposal.

The main difference of theoretical importance between this solution of the optimum combination of products and that shown in Fig. 3.9 is that now the iso-revenue line is not a tangent to the production-possibility boundary at the optimum point. Instead the optimum is found

at a corner point or basic feasible solution. Whereas when the boundary is a smooth curve even a slight change in the root: grain price ratio causes a change in the optimum combination of these products, now point M in Fig. 3.10 is the optimum over quite a wide range of price ratios. Thus the slope of the iso-revenue line may vary from the slope of MN to the slope of LM without changing the optimum.

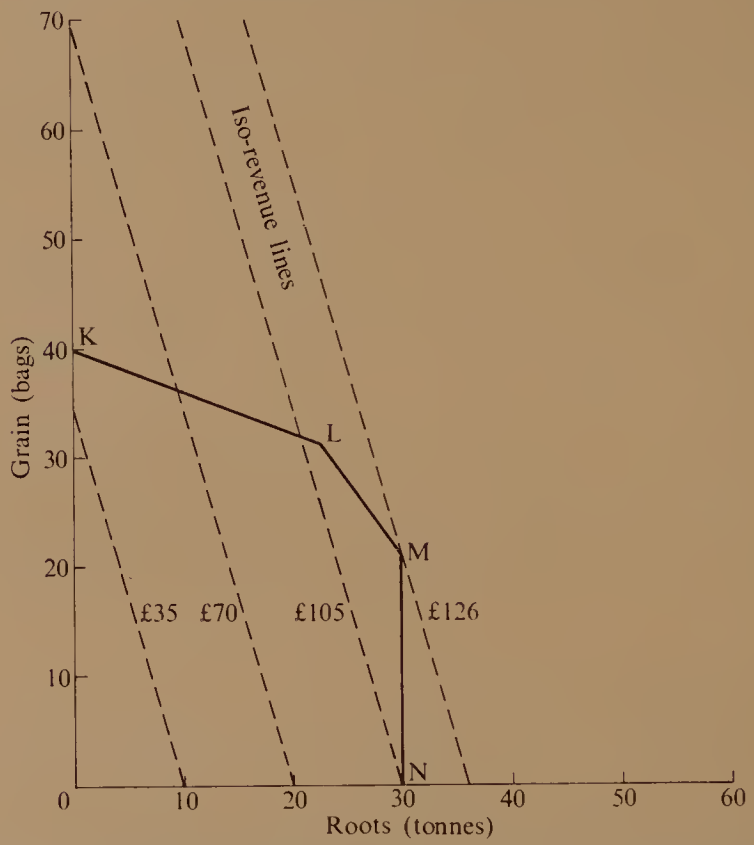


FIG. 3.10. The linear programming solution. As in Fig. 3.9 the maximum possible revenue is obtained where the highest valued iso-revenue line just touches the boundary (point M). However, this optimum now occurs at a corner rather than at a point of tangency.

The total revenue for each of the basic feasible solutions is gven in Table 3.9. These figures are obtained from Table 2.8 (p. 58) by apiplying the relevant prices to the physical products of roots and grain. This table bears out the fact that point M in Fig. 3.10 is the optimum combination of products. We can also see from this table that although the rate of product transformation does not *equal* the inverse price ratio of 3·5 at point M, nevertheless it does change from 1·4 to infinity, and 3·5 lies between these two rates. Point M is the optimum for any root:grain

price ratio from 1·4 to infinity. Since the price of roots is unlikely ever to be infinite, the optimum is unlikely to occur at point N. Some grain will always be found in the optimum solution no matter what the relative prices are. This is because grain is supplementary to roots between points N and M.

If we wish to *calculate* the economic optimum combination of products, all physical outputs must be evaluated in the same units, usually money. We then have a typical linear programming problem of maximizing the total returns (which are now financial rather than physical) subject

TABLE 3.9

Optimum product combination—comparison of corner points
(Inverse price ratio = £3.5/£1 = 3·5)

Point in Fig. 2.12	Total product roots (tonnes)	Total product grain (bags)	Total revenue (£)	Rate of product transformation (bags of grain per tonne of roots)
K	0	40	40	
				0·4
L	23	31	111.5	
				1·4
M	30	21	126	
				∞
N	30	0	105	

to a set of fixed constraints. In practice outputs must be evaluated in terms of gross margins, rather than total revenues per unit, because in linear programming we assume that for each activity all inputs and the output vary in strict proportion. Given constant prices for all variable hired inputs and the product output, the variable cost and the gross margin per unit of output also remain constant. Since fixed costs are assumed not to vary with output, by maximizing the total gross margin, profit is also maximized.

Note that the variable costs must be specific to the enterprise besides varying in proportion to the size of the enterprise. For farm planning purposes, in Britain at least, the following are defined as variable costs. For crops: fertilizer, seed, sprays, casual labour, and contract work specific to the crop. For livestock: concentrate feeds, veterinary expenses, marketing costs, and forage crop variable costs. In the present example the 'prices' of £1 per bag of grain and £3.50 per tonne of roots are really gross margins.

These unit gross margins are given in Table 3.10, together with the

constraint levels and input–output coefficients from Table 2.6 (p. 55). A solution may be found to this very simple illustrative example by the intuitive method used in the previous chapter. Thus the first step is to find out which activity will yield the maximum return (total gross margin) to the limited resources. It turns out that the maximum total gross margin that can be earned from a single activity is the £105 obtained when root production is expanded up to the limit set by the December

TABLE 3.10

The linear programming matrix

	Constraint level	Activity 1: grain production	Activity 2: root production
Gross margin per unit (£)		1	3.50
Land (hectares)	3	0·04	0·05
Total labour (units)	5	0·08	0·11
June labour (units)	1	0·025	0·01
December labour (units)	0·9	0	0·03

labour constraint. Thus we have moved from the basic feasible solution represented by the origin to the basic feasible solution at point N in Fig. 3.10, and Table 3.9. At this point 1·5 hectares of land, 1·7 units of total labour, and 0·7 units of June labour remain in disposal.

The next step is to investigate whether total gross margin can be increased by introducing a new activity into the plan. In this case it is immediately obvious that grain production can be introduced without displacing any of the root-crop activity because grain production does not compete with roots for the effective constraining resource, December labour. Of the other constraints, total labour now limits the level of grain production to 1·7/0·08 = 21 bags. The land in disposal would allow the production of 1·5/0·04 = 37 bags and June labour would allow 0·7/0·025 = 28 bags.

Thus we move to point M in Table 3.9 and Fig. 3.10. The new plan or combination of activities consists of 30 tonnes of roots and 21 bags of grain, which uses up all the total labour and December labour exactly and yields a total gross margin of £126, as we may see from Table 3.11. This now is the optimum plan. It is not possible to increase total gross margin further by any change in the combination of grain- and root-production activities, subject to the constraints of 5 units of total labour and 0·9 units of labour in December. The land and June labour constraints

are not effective in limiting the total gross margin since there is some of each of these resources in disposal.

To complete the analysis, the effects on the optimum gross margin of varying individual constraints may be estimated. In other words the shadow-prices may be determined. First let us consider the effect of relaxing the land or June labour constraints, which are not effective. Clearly it is not possible to increase total gross margin by increasing the quantities of land or June labour available because these constraints are not effective in limiting production and hence total gross margin.

TABLE 3.11

The economic optimum solution

	Grain production	Root production	Disposal	Total
Total gross margin (£)	21 × 1	+ 30 × 3.50	+ 0	= £126
Land (hectares)	21 × 0·04	+ 30 × 0·05	+ 0·65 × 1	= 3
Total labour (units)	21 × 0·08	+ 30 × 0·11	+ 0	= 5
June labour (units)	21 × 0·025	+ 30 × 0·01	+ 0·175 × 1	= 1
December labour (units)	21 × 0	+ 30 × 0·03	+ 0	= 0·9

From Fig. 2.12 (p. 57) it is clear that shifting the land constraint line outwards has no effect on the production-possibility boundary OKLMN. Shifting the June labour constraint line outwards would shift the segment KL of the boundary but it would not affect point M, which we now know is the optimum combination of activities. All this amounts to saying that the marginal products or shadow-prices of land and June labour are zero in this example.

The available resources of total labour and labour in December are effective constraints so that relaxing either of them will allow output and hence total gross margin to be expanded, and reducing either of them will have the opposite effect. Let us consider the effect of increasing total labour supply by 1 unit. This extra unit of labour cannot be used to increase root production because that is restricted to 30 tonnes by the December labour constraint. However it would allow grain production to be increased by 1/0·08 = 12·5 bags which would yield £12.5 extra gross margin. The effect may be illustrated in Figs 2.12 and 3.10 by shifting the line LM outwards by $\frac{1}{5}$ without changing its slope. In fact the land constraint would become effective before a whole extra unit of labour could be brought into use, but we are concerned with estimating the marginal product at the optimum point, M. The problem does not

arise if we consider a reduction of the total labour supply by 1 unit, and since this would reduce total gross margin by £12.5 this is the appropriate estimate of the shadow-price of total labour.

An extra unit of labour in December would allow root production to increase by 1/0·03 = 33·3 tonnes, yielding an additional gross margin of 33·3 × £3.5 = £116.7 but it would necessitate a reduction in grain output to release the necessary total labour. Thus 33·3 tonnes of roots requires 33·3 × 0·11 = 3·7 units of total labour. To release this, grain production must fall by 3·7/0·08 = 46·2 bags so that a gross margin of

TABLE 3.12

Product exhaustion under constant returns to scale

	Return to December labour (£)		Return to total labour (£)		Gross margin (£)
Grain production per bag	0 × 70.5	+	0.08 × 12.5	=	1
Root production per tonne	0.03 × 70.5	+	0.11 × 12.5	=	3.5
Whole plan	0.90 × 70.5	+	5 × 12.5	=	126

£46.2 must be forgone. The net result of increasing December labour by 1 unit is an increase in total gross margin of £116.7 − £46.2 = £70.5. This may be checked by an appropriate shift of the vertical line MN in Figs 2.12 and 3.10. Thus the shadow prices of total labour and December labour are estimated to be £12.5 and £70.5 respectively.

Since in linear programming we assume constant returns to scale, it would be consistent with our earlier theory if, when factors are paid according to their marginal products total output or total gross margin is accounted for exactly. This is in fact the case both for individual activities and for the whole plan as Table 3.12 shows.

Summary

Given a system of production under fixed constraints and/or decreasing returns to scale, if we assume conditions of pure competition so that unit prices for inputs and outputs remain constant, the following conditions are necessary and sufficient for profit maximization:

1. *For variable (unconstrained) factors, and smoothly diminishing marginal returns*:

(a) For a given single factor and a given single product *either* the value of the marginal product is everywhere less than the unit factor cost,

in which case that factor will not be used in producing that product *or* the value of the marginal product should equal the unit factor cost.
(b) It follows that for any two variable factors and a single product the rate of technical substitution should equal the inverse factor price ratio.
(c) It also follows from rule 1(a) that for any two products the rate of product transformation should equal the inverse product price ratio.

2. *For fixed factors with alternative uses*:
(a) For a given fixed factor and a particular product *either* the value of the marginal product in that use is everywhere less than the opportunity cost (i.e. the value of the marginal product in the most valuable alternative use), in which case the factor should not be used, *or* the value of the marginal product should equal the opportunity cost per unit.
(b) If some of the factor remains in disposal then the value of the marginal product and the opportunity cost per unit will be zero.

Mathematical appendix

The economic optimum for a single variable factor

We assume the production function

$$Y = f(X_1/X_2, X_3, \ldots, X_n)$$

has the characteristics described in Chapter 1 and the mathematical appendix, X_1 being variable and $X_2, X_3, \ldots, X_n$ being fixed.

Now let

p = price per unit of product Y,

r_1 = price per unit of variable factor X_1 (i.e. unit factor cost),

and

K = fixed cost which is constant.

Note that under pure competition all prices are assumed to be constant.

Hence

$$\text{total revenue } (TR) = pY,$$

$$\text{average revenue } (AR) = \frac{(TR)}{Y} = p,$$

$$\text{marginal revenue } (MR) = \frac{\mathrm{d}(TR)}{\mathrm{d}Y} = p$$

and the value of the marginal product $(VMP)_1 = p(MP)_1 = p\dfrac{dY}{dX_1}$. Also

$$\text{total cost } C = r_1 X_1 + K$$

$$\text{average cost } (AC) = \frac{C}{Y} = \frac{r_1 X_1 + K}{Y} = \frac{r_1}{(AP)_1} + \frac{K}{Y}$$

and

$$\text{marginal cost } (MC) = \frac{dC}{dY} = r_1 \frac{dX_1}{dY}.$$

Note that $\dfrac{dX_1}{dY}$ may be written as $1 \Big/ \dfrac{dY}{dX_1} = \dfrac{1}{(MP)_1}$,

so that

$$(MC) = r_1 \frac{dX_1}{dY} = \frac{r_1}{(MP)_1}$$

The economic optimum may now be expressed in several ways.

1. Profit is maximized when the extra revenue per unit of variable input is falling and just equals the extra cost:

$$(VMP)_1 = r_1, \tag{3.1}$$

i.e. the value of the marginal product equals unit factor cost.[9]

[9] Using the differential calculus:

$$\text{profit } \pi = pY - r_1 X_1 - K,$$

which is maximized when

$$\frac{d\pi}{dX_1} = 0 \quad \text{and} \quad \frac{d^2\pi}{dX_1^2} < 0.$$

If

$$\frac{d\pi}{dX_1} = p\frac{dY}{dX_1} - r_1 = 0, \text{ then } p\frac{dY}{dX_1} = r_1,$$

i.e. the value of the marginal product equals unit factor cost.

Now since

$$\frac{d^2\pi}{dX_1^2} = p\frac{d^2Y}{dX_1^2} \quad \text{and} \quad p > 0$$

then

$$\frac{d^2\pi}{dX_1^2} < 0 \text{ if and only if } \frac{d^2Y}{dX_1^2} < 0,$$

i.e. there must be diminishing marginal physical returns.

n.b. Here and in footnotes 10, 11, 12, and 13 we are ignoring the possibilities of (i) zero second derivatives, (ii) more than one local optimum, and (iii) corner optima.

2. Dividing both sides by r_1 gives

$$\frac{(VMP)_1}{r_1} = 1, \tag{3.2}$$

i.e. the value of the marginal product per £1 spent is unity.

3. Substituting $p \,.\, (MP)_1$ for $(VMP)_1$ in eqn (3.1) above and dividing both sides by p gives

$$(MP)_1 = \frac{r_1}{p}, \tag{3.3}$$

i.e. marginal physical product equals inverse (factor–product) price ratio, which in turn equals the price of input measured in units of physical product.

4. From Chapter 1 we have

$$\text{elasticity of response } E_1 = \frac{(MP)_1}{(AP)_1}$$

and

$$(AP)_1 = \frac{Y}{X_1} \text{ so } E_1 = (MP)_1 \frac{X_1}{Y}.$$

Substituting r_1/p for $(MP)_1$ (eqn (3.3) above) gives

$$E_1 = \frac{r_1 X_1}{pY} = \frac{\text{total cost of variable factor 1}}{\text{total revenue}} \tag{3.4}$$

i.e. the proportion of total revenue paid to the variable factor is equal to the elasticity of response.

5. Rearranging eqn (3.3) gives

$$p = \frac{r_1}{(MP)_1}.$$

But $p = (MR)$ and $r_1/(MP)_1 = (MC)$, hence, by substitution,

$$(MR) = (MC)$$

i.e. marginal revenue equals marginal cost. n.b. Since $(MP)_1$ must be falling and r_1 is constant, $(MC) = r_1/(MP)_1$ must be rising.

Two variable factors and the expansion path

Now, assume $Y = f(X_1, X_2/X_3, \ldots, X_n)$ but that Y is held constant at a given level Y^*, and let

r_1 = price per unit of variable factor 1 (unit factor cost 1)

r_2 = price per unit of variable factor 2 (unit factor cost 2).

Now:

$$(VMP)_1 = p(MP)_1 \text{ and } (VMP)_2 = p(MP)_2;$$
$$C = r_1X_1 + r_2X_2 + K,$$

thus C depends upon the quantities X_1 and X_2 in the factor–factor combination.

The least-cost combination may now be expressed in several ways.

1. The least-cost combination occurs where extra cost (negative gain) of substituting factor 2 for factor 1 just equals the extra saving (positive gain):

$$-\mathrm{d}X_2\, r_2 = \mathrm{d}X_1\, r_1. \tag{3.5}$$

2. Dividing both sides by $\mathrm{d}X_1\, r_2$ gives

$$-\frac{\mathrm{d}X_2}{\mathrm{d}X_1} = \frac{r_1}{r_2}$$

but

$$-\frac{\mathrm{d}X_2}{\mathrm{d}X_1} = (RTS)_{12}$$

(from Chapter 1). Hence

$$(RTS)_{12} = \frac{r_1}{r_2} \tag{3.6}$$

i.e. rate of technical substitution equals the inverse price ratio.[10]

[10] We wish to minimize $C = r_1X_1 + r_2X_2 + K$ subject to $Y^* = f(X_1,X_2)$ which is constant. Thus X_2 is dependent upon X_1 and C is minimized where

$$\frac{\mathrm{d}C}{\mathrm{d}X_1} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2C}{\mathrm{d}X_1{}^2} > 0.$$

When $$\frac{\mathrm{d}C}{\mathrm{d}X_1} = r_1 + r_2\frac{\mathrm{d}X_2}{\mathrm{d}X_1} = 0, \text{ then } r_2\frac{\mathrm{d}X_2}{\mathrm{d}X_1} = -r_1,$$

therefore

$$\frac{-\mathrm{d}X_2}{\mathrm{d}X_1} = (RTS)_{12} = \frac{r_1}{r_2}$$

i.e. rate of technical substitution equals the inverse price ratio.
Now,

$$\frac{\mathrm{d}^2C}{\mathrm{d}X_1{}^2} = r_2\frac{\mathrm{d}^2X_2}{\mathrm{d}X_1{}^2} = -r_2\frac{\mathrm{d}(RTS)_{12}}{\mathrm{d}X_1}.$$

Then since $r_2 > 0$, $\dfrac{\mathrm{d}^2C}{\mathrm{d}X_1{}^2} > 0$ if $\dfrac{\mathrm{d}(RTS)_{12}}{\mathrm{d}X_1} < 0$,

i.e. there must be diminishing rates of technical substitution.

3. But $(RTS)_{12}$ also equals $(MP)_1/(MP)_2$ (from Chapter 1). Hence by substitution in eqn (3.6) we have

$$\frac{(MP)_1}{(MP)_2} = \frac{r_1}{r_2}$$

or, multiplying both sides by $(MP)_2/r_1$,

$$\frac{(MP)_1}{r_1} = \frac{(MP)_2}{r_2}, \tag{3.7}$$

i.e. marginal product per £1 spent is the same for each factor.

The optimum rate of output

Now assume that output Y may be varied along the expansion path. Marginal cost is given by

$$(MC) = \frac{\mathrm{d}C}{\mathrm{d}Y}.$$

If both inputs 1 and 2 are varied

$$\mathrm{d}C = \mathrm{d}X_1\, r_1 + \mathrm{d}X_2\, r_2,$$

and

$$\mathrm{d}Y = \mathrm{d}X_1\,(MP)_1 + \mathrm{d}X_2\,(MP)_2.$$

In the first equation we may take r_1 or r_2 outside a bracket to give

$$\mathrm{d}C = r_1\left(\mathrm{d}X_1 + \mathrm{d}X_2\frac{r_2}{r_1}\right) = r_2\left(\mathrm{d}X_1\frac{r_1}{r_2} + \mathrm{d}X_2\right).$$

Similarly, in the second equation, we may take $(MP)_1$ or $(MP)_2$ outside a bracket to give

$$\mathrm{d}Y = (MP)_1\left(\mathrm{d}X_1 + \mathrm{d}X_2\frac{(MP)_2}{(MP)_1}\right) = (MP)_2\left(\mathrm{d}X_1\frac{(MP)_1}{(MP)_2} + \mathrm{d}X_2\right).$$

But at the least-cost combination

$$\frac{(MP)_1}{(MP)_2} = \frac{r_1}{r_2}$$

(from (3.7) above). So the terms inside the brackets are equal in both equations.

Hence:

$$(MC) = \frac{\mathrm{d}C}{\mathrm{d}Y} = \frac{r_1}{(MP)_1} = \frac{r_2}{(MP)_2}. \tag{3.8}$$

The economic optimum rate of output is found where

$$(MR) = (MC)$$

But $(MR) = p$ under pure competition so substituting this, and from (3.8) above, we obtain

$$p = \frac{r_1}{(MP)_1} = \frac{r_2}{(MP)_2}.$$

By rearrangement,

$$p(MP)_1 = r_1 \text{ and } p(MP)_2 = r_2, \tag{3.9}$$

i.e. the value of the marginal product equals unit factor cost for all variable inputs simultaneously.[11]

n.b. In order to arrive at these conditions for an optimum it is necessary that $(MP)_1$ and $(MP)_2$ should be diminishing and hence that $(MC) = r_1/(MP)_1 = r_2/(MP)_2$ should be rising. But we have seen (appendix to Chapter 1) that if inputs can be expanded to scale, and there are constant returns, $(MP)_1$ and $(MP)_2$ and marginal products

[11] The result from footnote 9 (p. 118) may be generalized to two or more variable factors as follows:

$$\pi = pY - r_1X_1 - r_2X_2 - K.$$

Profit π is maximized where $\left.\begin{aligned} \frac{\partial \pi}{\partial X_1} &= p\frac{\partial Y}{\partial X_1} - r_1 = 0 \\ \frac{\partial \pi}{\partial X_2} &= p\frac{\partial Y}{\partial X_2} - r_2 = 0 \end{aligned}\right\}$ simultaneously.

These equations imply that

$$p\frac{\partial Y}{\partial X_1} = r_1 \quad \text{and} \quad p\frac{\partial Y}{\partial X_2} = r_2$$

simultaneously. They may be expressed as a ratio to give the least-cost combination:

$$\frac{\partial Y}{\partial X_1} \Big/ \frac{\partial Y}{\partial X_2} = \frac{(MP)_1}{(MP)_2} = \frac{r_1}{r_2}.$$

The second-order conditions which are sufficient for a maximum to exist are:

$$\frac{\partial^2 \pi}{\partial X_1^2} = p\frac{\partial^2 Y}{\partial X_1^2} < 0 \quad \text{and} \quad \frac{\partial^2 \pi}{\partial X_2^2} = p \cdot \frac{\partial^2 Y}{\partial X_2^2} < 0,$$

i.e. there must be diminishing marginal returns to individual factors *and*

$$\frac{\partial^2 \pi}{\partial X_1^2}\frac{\partial^2 \pi}{\partial X_2^2} - \left(\frac{\partial^2 \pi}{\partial X_1 \partial X_2}\right)^2 = p^2\left\{\frac{\partial^2 Y}{\partial X_1^2}\frac{\partial^2 Y}{\partial X_2^2} - \left(\frac{\partial^2 Y}{\partial X_1 \partial X_2}\right)^2\right\} > 0,$$

i.e. the stationary point must not be a saddle point.

for all inputs remain constant. Hence under constant prices (MC) also remains constant and there is no optimum rate of output.

Constraints and shadow prices

If a factor is subject to contraint, the shadow price λ is defined as the value of the marginal product of the last available unit. Thus if factor 2 is constrained, $\lambda_2 = p(MP)_2$.[12]

Using the same argument as that used to derive eqn (3.4) we may write

$$E_2 = \frac{\lambda_2 X_2}{pY}.$$

Hence for factor 1 variable and factor 2 fixed we have

$$E_1 + E_2 = \frac{r_1 X_1}{pY} + \frac{\lambda_2 X_2}{pY}$$

or

$$pY(E_1 + E_2) = r_1 X_1 + \lambda_2 X_2.$$

[12] Problems of maximization (or minimization) subject to constraints may be solved by using differential calculus and Lagrange multipliers thus.
Assume X_2 is limited to level L so that $X_2 - L = 0$. Form the Lagrangian function:

$$\pi_\lambda = pY - r_1 X_1 - \lambda(X_2 - L),$$

where λ is an undetermined Lagrange multiplier (different from zero). π_λ and hence π are maximized where:

$$\left.\begin{aligned} \frac{\partial \pi_\lambda}{\partial X_1} &= p\frac{\partial Y}{\partial X_1} - r_1 = 0 \\ \frac{\partial \pi_\lambda}{\partial X_2} &= p\frac{\partial Y}{\partial X_2} - \lambda = 0 \\ \frac{\partial \pi_\lambda}{\partial \lambda} &= -X_2 + L = 0 \end{aligned}\right\} \text{simultaneously.}$$

These equations may be solved to give

$$p\frac{\partial Y}{\partial X_1} = r_1, \quad p\frac{\partial Y}{\partial X_2} = \lambda, \quad \text{and} \quad X_2 = L.$$

Now, it may be shown that λ is a measure of the marginal effect of relaxing the constraint, i.e. the shadow-price. We see above that it equals the value of the marginal product. The second-order condition, sufficient for a maximum, is in this case simply

$$\frac{\partial^2 Y}{\partial X_1^2} < 0.$$

Now if these 2 factors are the only factors involved in production and there are constant returns to scale,

$$E_1 + E_2 = e = 1.$$

Then:

$$pY = r_1 X_1 + \lambda_2 X_2,$$

i.e. if there are constant returns to scale and variable factors are rewarded according to their market prices while fixed factors are ascribed their shadow prices, the total revenue is exactly exhausted.

n.b. This is simply another version of Euler's theorem (see appendix to Chapter 1).

Choice of products

We now assume two products with separate production functions as specified in the mathematical appendix to Chapter 1, thus:

$$Y_1 = f_1(X_1/X_2, X_3, \ldots, X_n) \text{ and } Y_2 = f_2(X_1/X_{n+1}, X_{n+2}, \ldots, X_m).$$

Let p_1 = price per unit of Y_1;

p_2 = price per unit of Y_2.

Now

$$(VMP, Y_1) = p_1 \,.\, (MP, Y_1) \text{ and } (VMP, Y_2) = p_2 \,.\, (MP, Y_2);$$

$$(MC, Y_1) = \frac{r_1}{(MP, Y_1)} \qquad \text{and } (MC, Y_2) = \frac{r_1}{(MP, Y_2)}.$$

The profit-maximizing combination of products may now be expressed in several ways. (Note similarities with definition of least-cost combination of resources.)

1. Profit is maximized where the extra cost (opportunity cost) of reducing output of product 2 just equals the extra revenue from expanding output of product 1.

$$-\,\mathrm{d}Y_2 \,.\, p_2 = \mathrm{d}Y_1 \,.\, p_1. \tag{3.10}$$

2. Dividing both sides by $-\,\mathrm{d}Y_2 \,.\, p_1$ gives:

$$\frac{p_2}{p_1} = -\frac{\mathrm{d}Y_1}{\mathrm{d}Y_2} \text{ but } -\frac{\mathrm{d}Y_1}{\mathrm{d}Y_2} = (RPT)_{21}$$

(from Chapter 1).

Therefore

$$(RPT)_{21} = \frac{p_2}{p_1}, \tag{3.11}$$

i.e. rate of product transformation equals the inverse price ratio.

3. But

$$(RPT)_{21} = \frac{(MP, Y_1)}{(MP, Y_2)}$$

(from Chapter 1).

Hence by substituting in eqn (3.11) this gives

$$\frac{(MP, Y_1)}{(MP, Y_2)} = \frac{p_2}{p_1}. \tag{3.12}$$

Multiplying both sides by $p_1 \,.\, (MP, Y_2)$ gives

$$(MP, Y_1) \,.\, p_1 = (MP, Y_2) \,.\, p_2 \text{ or } (VMP, Y_1) = (VMP, Y_2),$$

i.e. the value of the marginal product per unit of variable factor should be equal in each enterprise: *principle of equimarginal returns.*[13]

[13] We wish to maximize total revenue $R = p_1 Y_1 + p_2 Y_2$ subject to the constraint $X_1 = L$. Thus we write $R_\lambda = p_1 Y_1 + p_2 Y_2 - \lambda(X_1 - L)$ and set

$$\left.\begin{aligned} \frac{\partial R_\lambda}{\partial Y_1} &= p_1 - \lambda \frac{\partial X_1}{\partial Y_1} = 0 \\ \frac{\partial R_\lambda}{\partial Y_2} &= p_2 - \lambda \frac{\partial X_1}{\partial Y_2} = 0 \\ \frac{\partial R_\lambda}{\partial \lambda} &= -X_1 + L = 0 \end{aligned}\right\} \text{simultaneously.}$$

From the first two equations we obtain:

$$\lambda = \frac{p_1}{\partial X_1/\partial Y_1} = \frac{p_2}{\partial X_1/\partial Y_2}.$$

But, by the inverse function rule,

$$\frac{\partial X_1}{\partial Y_1} = \frac{1}{(MP, Y_1)} \quad \text{and} \quad \frac{\partial X_1}{\partial Y_2} = \frac{1}{(MP, Y_2)},$$

therefore

$$\lambda = p(MP, Y_1) = p_2(MP, Y_2).$$

i.e. λ (the shadow-price of X_1) $= (VMP, Y_1) = (VMP, Y_2)$.

The second-order conditions require that

$$\frac{\mathrm{d}^2 Y_1}{\mathrm{d} Y_2^2} = \frac{\mathrm{d}(RPT)_{21}}{\mathrm{d} Y_2} > 0,$$

therefore the rate of product transformation must be increasing.

4. Since

$$(MC, Y_1) = \frac{r_1}{(MP, Y_1)} \quad \text{and} \quad (MC, Y_2) = \frac{r_1}{(MP, Y_2)}$$

$$\frac{(MC, Y_2)}{(MC, Y_1)} = \frac{r_1}{(MP, Y_2)} \cdot \frac{(MP, Y_1)}{r_1} = \frac{(MP, Y_1)}{(MP, Y_2)}.$$

Substituting in eqn (3.12) above gives:

$$\frac{(MC, Y_2)}{(MC, Y_1)} = \frac{p_2}{p_1},$$

i.e. the ratio of marginal costs equals the ratio of marginal revenues. Note that we are assuming that X_1 cannot be varied so (MC, Y_2) cannot be equated with p_2 and (MC, Y_1) cannot be equated with p_1. Nevertheless, subject to this constraint, revenue is maximized where the ratios are made equal.

Linear production relationships

The problem of finding the economic optimum combination of alternative activities yielding different products, when there are fixed constraints and all relationship are linear, may be solved by the methods described in Chapter 2 for finding the point of maximum output. The only difference in the formulation of the problem is that now we have an objective function in terms of the product values, generally the gross margins per unit of each activity. Thus if there are 3 real activities with gross margins per unit of C_1, C_2, and C_3 and 2 disposal activities, the objective function to be maximized is:

$$Z = C_1X_1 + C_2X_2 + C_3X_3 + 0X_4 + 0X_5.$$

This necessitates minor modifications in the revised simplex method. Thus in stage 2 the marginal products of the factors of production are estimated in terms of the product gross margins. For example if activities 2 and 3 are already in the basis then w_1 and w_2 are estimated by solving

$$a_{12} \cdot w_1 + a_{22} \cdot w_2 = C_2$$

and

$$a_{13} \cdot w_1 + a_{23} \cdot w_2 = C_3.$$

In stage 3 the net gain per unit of each of the activities *excluded* from the current BFS is the gross margin for that activity minus the opportunity

costs of the factor inputs. For instance if activity 1 is exluded from the BFS when there are 2 constraints we have:

Net gain per unit of activity $1 = C_1 - a_{11} \cdot w_1 - a_{21} \cdot w_2$.

Example. Again we illustrate with the problem set out in Tables 2.6 and 2.10. The objective function is now

$$\text{Maximize } Z = 1X_1 + 3{\cdot}50X_2 + 0K_3 + 0X_4 + 0X_5 + 0X_6.$$

1. The initial BFS representing the origin consists of

$X_3 = 3$ hectares land in disposal;
$X_4 = 5$ units total labour in disposal;
$X_5 = 1$ unit of June labour in disposal;
$X_6 = 0{\cdot}9$ units of December labour in disposal.

} BFS no. 1 in Table 2.10

2. At this stage (*MP*) of all 4 resources is zero.
3. The net gain per bag of grain production is therefore £1 (minus zero opportunity cost). The net gain per tonne of roots is £3.50 (minus zero opportunity cost). We therefore introduce root production into the BFS.
4. The maximum feasible level of root production is given by

$$X_2 = \min\left(\frac{3}{0{\cdot}05}, \frac{5}{0{\cdot}11}, \frac{1}{0{\cdot}01}, \frac{0{\cdot}9}{0{\cdot}03}\right)$$

$$= \min\left(\begin{array}{cccc} 60, & 45{\cdot}5 & 100 & 30) \\ \text{Land,} & \text{total} & \text{June} & \text{December} \\ & \text{labour,} & \text{labour,} & \text{labour,} \end{array}\right.$$

Hence the fourth constraint, December labour, is effective so December labour disposal activity is eliminated from the BFS.

5. The new BFS consists of;

$X_3 = 1{\cdot}5$ hectares land in disposal
$X_4 = 1{\cdot}7$ units total labour in disposal
$X_5 = 0{\cdot}7$ units of June labour in disposal

} BFS no. 5 in Table 2.10

$X_2 = 30$ tonnes of roots
$Z = 30 \times £3.50 = £105.$

End of iteration

2. We find (*MP*) for each of the factors of production by solving:

$$\begin{aligned} 1w_1 &+ 0w_2 &+ 0w_3 &+ 0w_4 &= 0 \\ 0w_1 &+ 1w_2 &+ 0w_3 &+ 0w_4 &= 0 \\ 0w_1 &+ 0w_2 &+ 1w_3 &+ 0w_4 &= 0 \\ 0{\cdot}05w_1 &+ 0{\cdot}11w_2 &+ 0{\cdot}01w_3 &+ 0{\cdot}03w_4 &= 3{\cdot}50. \end{aligned}$$

It is immediately obvious from the first 3 equations that

$$w_1 = w_2 = w_3 = 0.$$

Indeed this should be apparent without writing out the equations since the first 3 resources are in disposal, so the marginal products per unit of each must be zero. The fourth equation reduces to

$$0{\cdot}03w_4 = 3{\cdot}50.$$

Hence $w_4 = 116{\cdot}66$.

3. The net gain per unit of grain production is therefore

$$£\{1 - 0(116{\cdot}66)\} = £1.$$

Hence total gross margin can be increased by introducing grain production.

4. The rates of substitution p_{21}, p_{31}, p_{41}, and p_{51} are found by solving

$$\begin{aligned} 0{\cdot}05p_{21} + 1p_{31} + 0p_{41} + 0p_{51} &= 0{\cdot}04 \\ 0{\cdot}11p_{21} + 0p_{31} + 1p_{41} + 0p_{51} &= 0{\cdot}08 \\ 0{\cdot}01p_{21} + 0p_{31} + 0p_{41} + 1p_{51} &= 0{\cdot}025 \\ 0{\cdot}03p_{21} + 0p_{31} + 0p_{41} + 0p_{51} &= 0, \end{aligned}$$

which gives the result

$$p_{21} = 0, p_{31} = 0{\cdot}04, p_{41} = 0{\cdot}08, p_{51} = 0{\cdot}025.$$

Hence

$$\begin{aligned} X_1 &= \min \left(\frac{1{\cdot}5}{0{\cdot}04}, \quad \frac{1{\cdot}7}{0{\cdot}08}, \quad \frac{0{\cdot}7}{0{\cdot}025}, \quad \frac{30}{0} \right) \\ &= \min \ (37{\cdot}5, \quad 21, \quad 28, \quad \infty) \\ &\qquad \text{land,} \quad \text{total labour,} \quad \text{June labour,} \quad \text{roots} \end{aligned}$$

Thus total labour disposal is the activity to be eliminated.

5. The new BFS consists of

$X_3 = 0{\cdot}65$ hectares of land in disposal
$X_1 = 21$ bags of grain
$X_5 = 0{\cdot}175$ units of June labour in disposal
} BFS no. 4 in Table 2.10
$X_2 = 30$ tonnes of roots
$Z = 21 \times £1 + 30 \times £3.50 = £126.$

End of second iteration

2. We find (*MP*) for each of the factors of production by solving

$$\begin{aligned} 1w_1 + 0w_2 + 0w_3 + 0w_4 &= 0 \\ 0{\cdot}04w_1 + 0{\cdot}08w_2 + 0{\cdot}025w_3 + 0w_4 &= 1{\cdot}00 \\ 0w_1 + 0w_2 + 1w_3 + 0w_4 &= 0 \\ 0{\cdot}05w_1 + 0{\cdot}11w_2 + 0{\cdot}01w_3 + 0{\cdot}03w_4 &= 3{\cdot}50. \end{aligned}$$

Hence

$$w_1 = 0;\ w_2 = 12{\cdot}5;\ w_3 = 0;\ w_4 = 70{\cdot}85.$$

There are no alternative activities to consider so the last-mentioned BFS must be the optimum.

Suggestions for further reading

BAUMOL, W. J. (1972). *Economic theory and operations analysis* (3rd ed.). Prentice-Hall, Englewood Cliffs, New Jersey.

LANCASTER, K. (1969). *Introduction to modern microeconomics*. Rand McNally, Chicago.

Also see suggestions for further reading following Chapters 1 and 2.

4. Decision making under risk and uncertainty

Variation in agricultural production

So far we have discussed management decisions as though production relationships were exact, all prices were constant, and the farmer had perfect knowledge of both. We now consider the effects of introducing risk and uncertainty into the analysis.

Production relationships are not exact because the farmer does not have perfect control of all the environmental factors which influence crop and livestock yields. Hence, whilst some of the variation in yields may be due to differences in the combination of inputs used, there is always a certain amount of variation due to uncontrolled factors. This residual variation may be viewed as chance variation; the result of good or bad luck. Under a primitive system of agriculture the cultivator controls very few inputs other than labour. Scientific developments enable farmers to take more and more control of the production process. As a result not only are yields increased but also the variation in yields is reduced. Thus the use of artificial fertilizers reduces the dependence of crop yields on natural variations in soil fertility; the use of insect and disease control measures reduces the variation in the effects of these pests on crop yields; while irrigation and drainage reduce the effects of variations in rainfall. For livestock which are housed, and fed balanced rations, the farmer probably has even greater control over the environment than for most crops.

Nevertheless, even in the most scientifically advanced production processes there is inevitably an element of luck in the results obtained. Indeed, pure scientists working under the most strictly controlled laboratory conditions still find some chance variation in the results of their experiments. Therefore it is impossible to predict the outcome of any productive process exactly; there is always some chance variation and therefore some uncertainty about the precise outcome of a particular productive activity over a particular period.

Furthermore the health of the farmer himself and the availability of his fixed resources cannot be predicted with certainty. Thus the illness, injury, or sudden death of the farmer, and floods, fires, or machinery

breakdowns are all unpredictable occurrences which could have important effects on the production process. Here again knowledge of the future must be imperfect.

Price variation

Although individuals, by their own actions, may have no perceptible influence on market prices under pure competition, the combined decisions of a large number of individuals may bring about sizeable price changes. For example an individual farmer may plant a new variety of wheat and increase his production without affecting the market price. But as large numbers of other farmers adopt this new variety, total wheat production will be increased so that prices will be affected. This is but one example of the way new production techniques can lower costs, increase total output, and thus lead to a fall in product price. In a similar manner the decision of one family to buy a turkey to celebrate Christmas would have little effect upon the market price. But if many families buy turkeys for the same occasion there will be a marked temporary rise in price.

There are many other forces which contribute to price fluctuations for farm products. They operate by causing changes in either the quantity supplied for sale or the quantity desired for purchase or both. A general increase in the quantity offered for sale is likely to result in a fall in price, other things being equal; conversely a general increase in the quantity desired for purchase is likely to result in a price rise.

Four types of price movement may be distinguished. First, there are long-term trends normally lasting over a period of several years or longer. These are caused by changes in population, incomes, tastes, and techniques of production. Secondly, some prices move in cycles; alternating rises and falls every few years. To some extent these cycles may be self-perpetuating. The third type of price movement is a seasonal fluctuation in prices which takes place within a year and may result from the seasonal pattern of production, prices tending to fall at harvest time. Alternatively a seasonal price change may be caused by variation in the quantity desired by consumers. Thus the price of poultry and other meat rises at the time of major festivals during the year. Finally, there are irregular movements, many of which may be quite unpredictable. They may be caused by a drought, a sudden outbreak of pests or diseases of crops or livestock, or they may result from an unexpected social change, such as the removal of trading restrictions on agricultural products.

From the farmer's point of view there is an important distinction between those price changes which *can* be predicted and those which cannot. He can allow for predictable price changes in making his management decisions but unpredictable changes are a cause of risk and uncertainty. Most seasonal price fluctuations are predictable, so a farmer may plan the timing of his production to coincide with periods of high prices. By harvesting and selling his crops early, before most other farmers, he might avoid the low prices at harvest time. Alternatively he may store his crops at harvest time to sell or consume them later when market prices have risen. The production of poultry and other livestock may be timed so that they are ready for slaughter and sale just before Christmas. Naturally, if large numbers of farmers change their pattern of production so as to take advantage of seasonal price increases, this in itself will tend to reduce these price increases, so the benefits of adjusting to these seasonal increases will be small. However, it is most unlikely that the seasonal price differences will be eliminated altogether because costs are involved in early production, storage, or careful timing of output. There would be no incentive for farmers to adjust if the differences in product prices were insufficient to cover the extra costs.

Price trends and cycles are less easy to predict, although careful study of price movements in the past can provide a basis for estimating how they will move in the future. However, irregular price changes are by their very nature largely unpredictable. Thus as far as the farmer is concerned input and output prices are likely to be subject to chance variation.

Even where prices for agricultural inputs and outputs are regulated by the Government, there is generally some variation over time. Although short-run irregularities may be eliminated, it is likely that price levels would have to be adjusted in response to long-lasting changes in supply or demand. For instance, if a technological innovation results in increased quantities of a given commodity being produced at a given price, unsold surpluses are likely to arise unless the price is lowered so that consumers will buy more. It is possible to maintain a constant price to farmers by paying a subsidy while the consumers' price is lowered but the Government may well be unable or unwilling to pay a subsidy over a very long period. Alternatively, if a deficit should arise, for instance because of an influx of refugees and hence an increase in the quantity of farm produce demanded, then attempts to hold prices down at a constant level are likely to be thwarted by the development of illicit trading. Furthermore, government actions are also influenced by social and political considerations and therefore cannot always be

predicted, particularly between elections and changes in the governing party or group.

Probability and expectations

Because of the variation in agricultural production and prices no decision can have a unique predetermined outcome; rather, there will be a range of possible outcomes. For instance, the marginal product of the fourth unit of labour used with a fixed quantity of other factors might possibly be anything from, say, 3 bags of grain up to 9 bags of grain. Now, some of the possible outcomes may be more likely to occur

TABLE 4.1

Probabilties of different numbers of lambs per ewe

Outcome: lambs per ewe	Frequency: number of ewes bearing that number of lambs	Probability: relative frequency of that number of lambs
0	2	2/200 = 0·01
1	100	100/200 = 0·50
2	80	80/200 = 0·40
3	18	18/200 = 0·09
Total	200	200/200 = 1·00

than others. Thus the likelihood of the marginal product of the fourth unit of labour being between 5 and 6 bags of grain may be greater than the likelihood of it being between 4 and 5 bags of grain. If the likelihood of a particular outcome can be expressed by a number, it is known as the 'probability' of that outcome.

For a given productive activity the probability of a particular outcome may be estimated on the basis of past experience of the same activity in terms of the relative frequency of that outcome. In other words, the probability of a particular outcome is given by dividing the number of times that outcome occurred by the total number of times the activity took place or the total number of 'trials' of that activity. This is best illustrated by an example where there is a limited number of outcomes. For instance the number of lambs born per ewe in a single lambing season is such a discrete variable. The whole range of possible outcomes might be 0, 1, 2, or 3 lambs. An outcome of, say, $1\frac{1}{2}$ lambs is quite impossible. (In contrast, the yield of a crop is continuously variable so that, strictly speaking, the number of possible outcomes is infinite.) Table 4.1 gives the result of one year's lamb production for a flock of

200 ewes. If it can be safely assumed that the conditions affecting ewe fertility will be the same, the probabilities given in the last column may be used as a guide to the number of lambs a typical ewe might bear in future years.

However, there are two reasons for doubting the reliability of this approach. One is the question of whether conditions affecting ewe fertility are likely to be constant between individual ewes and from year to year. If this is not the case the results obtained with one flock of ewes in a particular season may be a very poor guide as to the performance of different ewes in different seasons. In practice it is unlikely that conditions would be constant. Therefore we should ask whether conditions under which the probabilities were estimated and the conditions under which they will be applied are sufficiently similar to justify this approach.

The other reason for doubting the reliability of this approach is that it is based on a limited number of observations, namely 200, in this case. Clearly there must always be a limit on the number of observations made and as a result the probability estimates can never be exact. The so-called 'law of large numbers' tells us that as the number of times an activity or trial is repeated grows larger and larger, the relative frequency with which a given outcome occurs will approach nearer and nearer to the probability of that outcome for a single trial. As a result the calculated relative frequencies can be viewed only as estimates of the actual probabilities.

Various important properties of probabilities may be illustrated by the figures given in Table 4.1. First, it may be noted that all probabilities lie between 0 and 1. The probability of an outcome which is impossible such as a ewe bearing a negative number of lambs or bearing, say, 10 lambs, is zero. Alternatively, if a particular outcome was certain to occur it would have a probability of 1. Generally, each one of a set of possible outcomes will have a probability between these two limits, as shown in the table. Secondly, the sum of the probabilities of all possible outcomes is equal to 1. This is because we are certain that one of these outcomes will occur. It follows from this second property that the probability of a particular outcome *not* occurring is equal to 1 minus the probability that it will occur. From the table it appears that the probability of a ewe not bearing triplets is $1 - 0{\cdot}09 = 0{\cdot}91$.

The word 'event' may be used to denote a group of possible outcomes. For example, the event of a multiple birth, i.e. a ewe bearing more than one lamb, is made up of the outcomes of twin births and triplet births. As one might expect, the probability of a given event occurring is the

sum of the probabilities of the outcomes making up that event. From Table 4.1 we can estimate the probability of a multiple birth as 0·40 + 0·09 = 0·49. Events may or may not be mutually exclusive. Thus the event of 'one or two lambs born' and the event 'multiple birth' are not mutually exclusive because the outcome of twins is common to both. For events which *are* mutually exclusive, the probability of one or other occurring is the sum of their separate probabilities. If they are not mutually exclusive the probability of one event or the other or both is the sum of their separate probabilities minus the probability that both happen together. In our example, the probability of 'one or two lambs born' is 0·50 + 0·40 = 0·90, and that of 'multiple birth' is 0·49, as we have seen. The probability of one or other or both of these events occurring is

$$0{\cdot}90 + 0{\cdot}49 - 0{\cdot}40 = 0{\cdot}99.$$

A situation where the decision maker knows all the possible outcomes of each action and can estimate the probability of each one occurring is said to be a risk situation. This is in contrast to the imaginary situation of certainty, where each action is known to lead invariably to a specific outcome, and to the real-world situation of uncertainty where probabilities cannot be estimated objectively. The quantity that is subject to chance variation in a risk situation is known as a random variable. Thus we have implied that the number of lambs a ewe may bear is a random variable. In practice, outcomes for many decisions cannot be predicted exactly and there is some justification for treating weather outcomes at specific seasons, and crop or livestock yields for given levels of inputs, as random variables.

The complete set of outcomes and their probabilities given in Table 4.1 represents a probability distribution which may be represented diagrammatically as in Fig. 4.1(a). This is a bar graph, or histogram, with the area of each bar representing the probability of that outcome. The total area of the histogram equals one. The number of lambs born per ewe is a discrete variable in that the number of outcomes is limited and countable. In fact in this case there are only four outcomes. In contrast, continuous variables can assume all values on a continuous scale so that in effect the number of possible outcomes in infinite. Continuous variables are usually measurements such as weight, time, length, volume, or price. Thus milk yield per cow or grain per hectare are continuous variables. Fig. 4.1(c) represents the frequency distribution of a continuous variable, which is generally a smooth curve. Again, the total area under the curve is equal to 1. In this case it is not meaningful

to give the probability of a single specific outcome, such as a yield of 3000 l (litres) exactly. However, the probability of an event, say a yield of between 3000 l and 3500 l can be estimated. Fig. 4.1(b) represents the case of a discrete variable with a large number of possible outcomes.

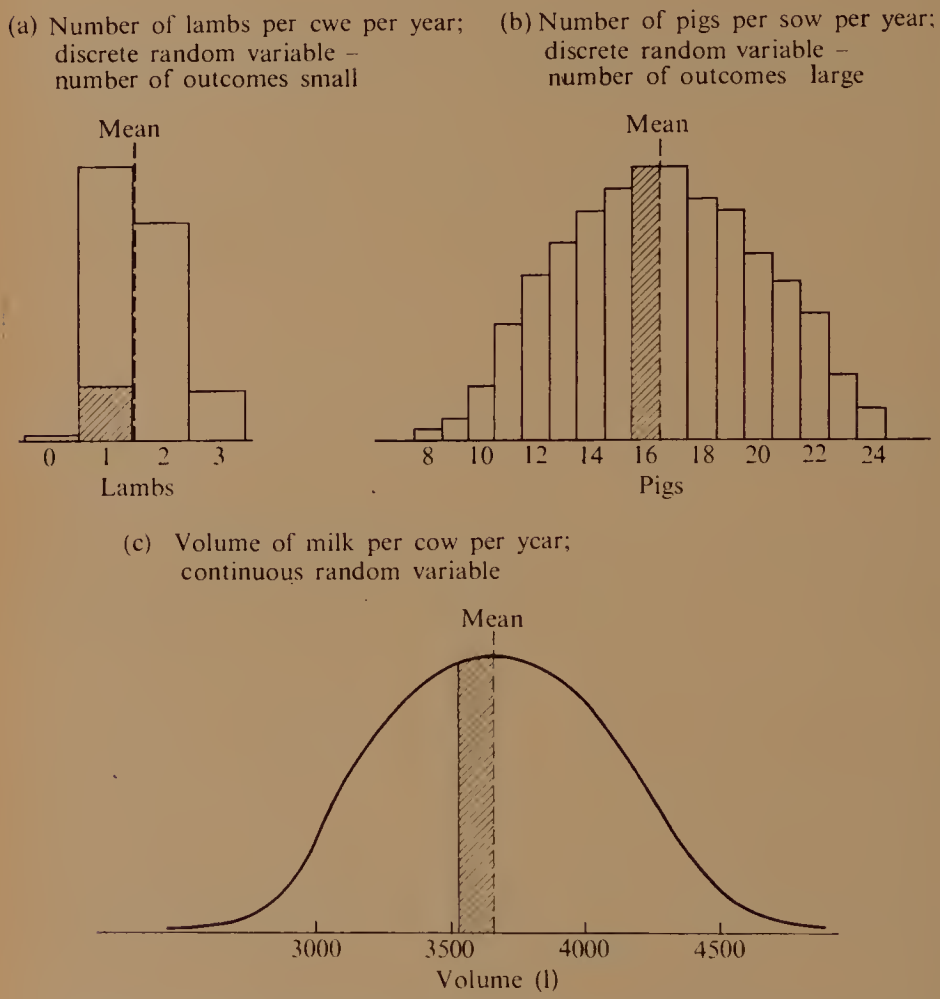


FIG. 4.1. Probability distributions. In each case the horizontal scale measures the variable with which we are concerned, (a) number of lambs per ewe, (b) number of pigs per sow, or (c) litres of milk per cow. Over a given interval on this scale, the area vertically above represents the probability (or relative frequency) of occurrence. For discrete random variables, as the number of possible outcomes increases the distribution becomes more nearly continuous.

It is clearly more closely similar to a continuous distribution than Fig. 4.1(a). Where the number of possible outcomes for a discrete variable becomes very large, as for example in the case of the number of eggs

produced per hen, the distribution may be closely approximated by a continuous distribution.

The 'expected value' of a discrete random variable is obtained by multiplying each possible outcome by the probability of its occurrence and then summing these products for all possible outcomes. Thus for the example given in Table 4.1 the *expected* number of lambs per ewe is calculated as

$$(0 \times 0{\cdot}01) + (1 \times 0{\cdot}50) + (2 \times 0{\cdot}40) + (3 \times 0{\cdot}09) = 1{\cdot}57.$$

How is this expected value to be interpreted? Clearly we cannot expect a single ewe to give birth to 1·57 lambs. Indeed, the probability of that outcome is zero; Table 4.1 shows that the most probable outcome for a single ewe is that she bears one lamb. No, the expected value represents the *average* number, or arithmetic mean, over a large number of trials. Thus from the results in Table 4.1 the total number of lambs born is calculated to be 314. The mean number per ewe is therefore 314/200 = 1·57. It may be argued logically that if *on average* each ewe bore 1·57 lambs in the past, the same result may be *expected on average* under identical conditions in the future.[1]

For a continuous variable, such as grain yield, the number of possible outcomes is infinite and the probability of an individual outcome is not defined so we cannot calculate the expected value by means of the first approach given above. Nevertheless the arithmetic mean of the frequency distribution is the expected value. Thus the expected value for the yield of a given grain crop under specific conditions is the mean yield obtained under similar conditions in the past.

Thus under conditions of risk the result of any action may be expressed in terms of expected values. For example in Table 1.1 (p. 7), the figures given for the total products in tonnes of potatoes per hectare are expected values. The yield of a particular potato crop on a particular plot in a particular season with the first level of application of nitrogen, phosphate, and potash is likely to differ from the 34·6 tonnes per hectare given in the table. However, over many seasons and many plots of potatoes on

[1] Strictly speaking, in order to measure probabilities and expected values exactly, the entire population (in this case all ewes of the given breed under given conditions at all times) would have to be observed. Unbiased estimates of expected values can be obtained from a small group of individuals, known as a sample, provided that the sample is selected at random, which means that every possible sample of a given size from the whole population has an equal chance of selection. Clearly this is not the case if the farmer has only the results from his own flock in the previous season to provide the estimates. However, for most business decisions, estimates from non-random or unrepresentative samples will be the best that are available.

fen soil, the average yield for that level of fertilizer is expected to be close to the figure given. Similarly all the estimates of product outputs, rates of substitution, and prices given in earlier chapters should be viewed as expected values.

Now, in a situation of risk where profits are not uniquely determined, a particular productive activity carried on at a given rate will have a range of possible financial outcomes which *may* vary from very large gains to substantial losses. Thus it is impossible to make decisions on the basis of maximizing *actual* profit. Actual profit is subject to risk which is outside the farmer's control. However, in a situation of risk it is possible to make decisions with the objective of maximizing *expected* profit or expected monetary value (EMV) as it is sometimes called. Decisions reached on the basis of this criterion may result in losses on occasions but over many seasons, that is if each decision is repeated many times, greater than average gains will eventually offset lower than average gains and losses, assuming the distribution of gains and losses is stable over time. As a result long-run average profit is maximized by always choosing the alternative having the highest EMV provided there is no carry-over effect from one season to the next. Furthermore, provided that the probability distribution of outcomes is known, an EMV is an *objective* measure of value. It does not vary from one decision maker to another according to his subjective attitudes and judgements.

Conditional probability

For a straightforward gamble, say on the toss of a coin, the financial outcome is a simple event; it does not depend upon any intermediate events after the placing of the bet. In contrast, the financial outcome of productive activity is a compound event; it depends upon a whole series of intermediate events such as whether there is a late frost in March, whether a machine breaks down or a worker falls ill at a critical period, and whether the market price is the same as that obtained last year. As a result the estimation of the EMV is more complicated than for a simple gamble, and care is needed to avoid errors.

Some aspects may be illustrated by considering the relationship between physical yield, product price, and revenue, when both physical yield and product price are subject to risk. More specifically we may consider the relationship between the number of lambs born per ewe, the price per lamb, and the revenue per ewe. We assume the probabilities of the various possible numbers of lambs born per ewe are as set out in Table 4.1. Now let us assume that lambs may sell at either of two grades, grade I at a price of £10 or grade 2 at a price of £8. The occurrence

of a given financial outcome (revenue per ewe) is dependent on the joint occurrence of a given yield and a given price outcome. For example, the occurrence of the outcome £8 per ewe is the result of the joint occurrence of the outcome (a) 1 lamb born and (b) a price of £8 obtained. The probability of this joint occurrence, known as the joint probability,

TABLE 4.2

Expected revenue per ewe

Compound event	Revenue (£)	Joint probability		Expectation (£)
No lambs	0		0·01	0
One lamb:				
grade 2	8	0·5 × 0·6	= 0·3	2.4
grade 1	10	0·5 × 0·4	= 0·2	2.0
Two lambs:				
both grade 2	16	0·4 × 0·6 × 0·6	= 0·144	2.304
one grade 1	18	0·4 × 0·6 × 0·4 × 2	= 0·192	3.456
both grade 1	20	0·4 × 0·4 × 0·4	= 0·064	1.280
Three lambs:				
all grade 2	24	0·09 × 0·6 × 0·6 × 0·6	= 0·019	0.467
one grade 1	26	0·09 × 0·6 × 0·6 × 0·4 × 3	= 0·039	1.011
two grade 1	28	0·09 × 0·6 × 0·4 × 0·4 × 3	= 0·026	0.726
all grade 1	30	0·09 × 0·4 × 0·4 × 0·4	= 0·006	0.172
		Total	= 1·000	£13.816

Notes

1. For a single lamb there are only two possible 'revenue per ewe' outcomes, either £8 or £10. Joint probabilities of two events, e.g. one lamb born *and* price of £8, calculated as described in text.
2. For twin lambs there are three possible outcomes, namely both sell for £8, one sells for £8 and the other for £10, and both sell for £10. The joint probability of three events is the product of all three probabilities, e.g. the probability of twins both being sold for £8 is given by the probability of a twin birth multiplied by the probability of one selling at £8, then multiplied by the probability of a second selling at £8. Note however, that there are two ways in which one may sell for £8 and one for £10, that is *either* the first *or* the second may obtain the higher price. Hence the joint probability of one selling for £8 and the other for £10 is doubled.
3. For triplets, there are three ways in which one can sell at a different price from the other two.

is given by the probability of bearing 1 lamb multiplied by the probability of a price of £8, *given that it was a single lamb*. This probability of a particular price outcome given that a particular yield outcome has occurred is known as the *conditional probability*. Thus let us say that

the conditional probability of a price of £8 being obtained given that a single lamb is born is 0·6, while, from Table 4.1, the probability of a ewe bearing a single lamb is 0·5. The probability of a revenue per ewe of £8 is therefore $0{\cdot}5 \times 0{\cdot}6 = 0{\cdot}3$. This general rule of multiplication can easily be extended to the occurrence of more than two events. In fact, the joint probability of the occurrence of several successive events can be obtained by multiplying the probabilities of the individual events assuming in each case that all previous events have occurred. The calculation of joint probabilities for 3 and 4 events is illustrated in Table 4.2.

The simplest situation arises where events, in our example price per lamb and number of lambs born, are independent. This means that the probability of one event occurring is in no way affected by the occurrence or the non-occurrence of the other event. In this case the conditional probability of the second event is no different from the simple probability of its occurrence. Thus if the probability of obtaining a price per lamb of £8 is always 0·6 regardless of the number of lambs per ewe, then clearly it is *independent* of the number of lambs. Where events are independent so too are the expected values, which means that they may be multiplied, divided, added, or subtracted as though they were certain outcomes. If the probability of obtaining £8 per lamb is 0·6 and the probability of obtaining £10 is 0·4, then the expected price per lamb is

$$(0{\cdot}6 \times £8) + (0{\cdot}4 \times £10) = £8.8.$$

Therefore the expected revenue per ewe is simply the expected number of lambs per ewe multiplied by the expected price per lamb, as follows:

$$1{\cdot}57 \times £8.8 = £13.816.$$

As a check on this result the probabilities of all possible 'revenue per ewe' outcomes are calculated in Table 4.2. These joint probabilities are then used to calculate the expected revenue per ewe which is identical with the figure just obtained.

Where conditional probabilities of the second event differ according to the outcome of the first event; i.e. where events are *dependent*, then so too are the expected values. For instance, if we now assume that the probability of obtaining a price of £10 is 0·6 for a single lamb, 0·5 for a twin lamb, and 0·4 for one of triplets, this implies that the price obtained is dependent or conditional upon the number of lambs born per ewe, which means that expected price per lamb is dependent upon the number of lambs born. In fact there is some scientific evidence that twin and triplet lambs do not fatten as fast as single lambs do and hence may

fetch a lower price when sold. There is therefore some justification for our assumption.

Now in order to estimate the expected revenue per ewe we need to consider all possible outcomes and their relevant probabilities, the

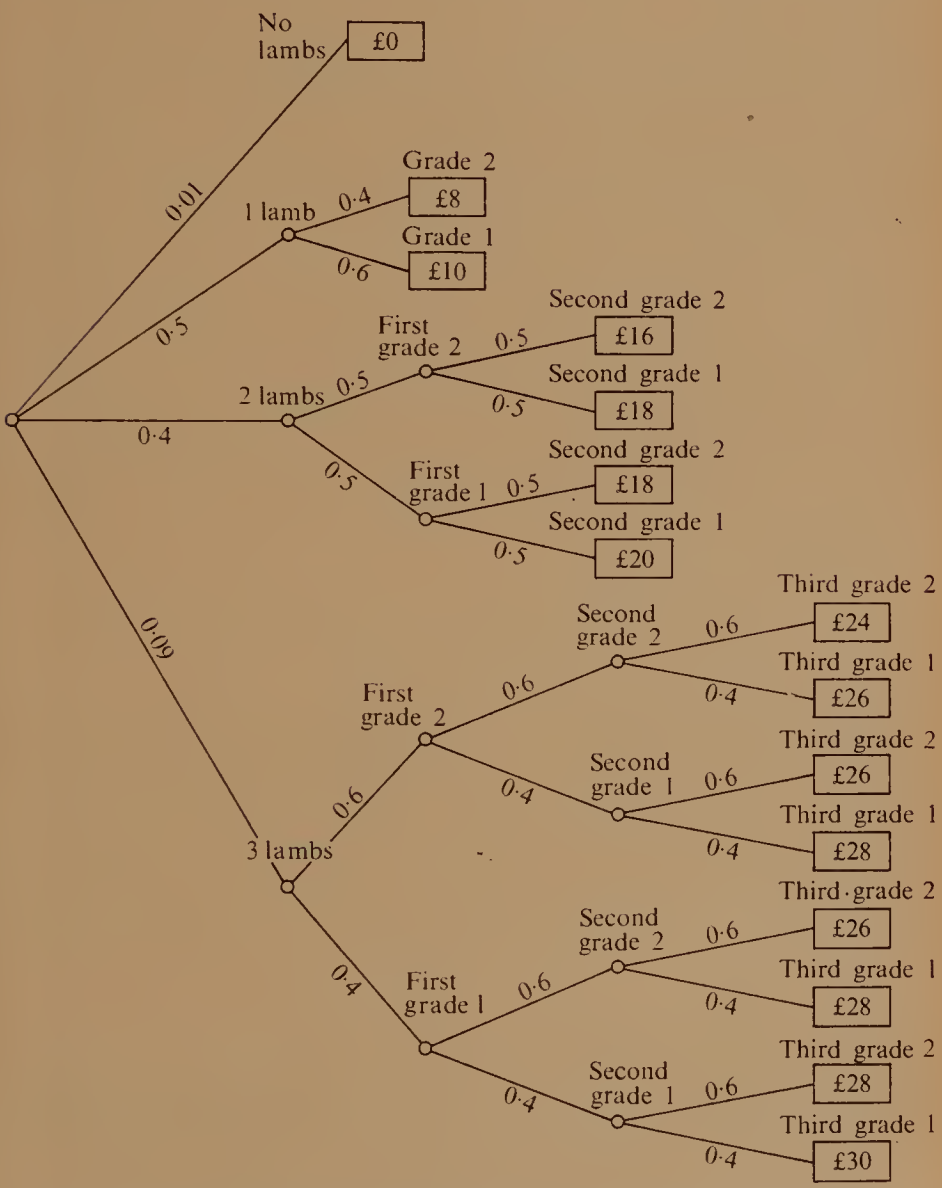


FIG. 4.2. Tree diagram of revenue per ewe. For compound events it is convenient to set out all possible outcomes and their respective probabilities in the form of a tree. Each node (or junction) represents a particular outcome, the final outcomes in terms of revenues appearing on the right. On the branch leading to each outcome node is written the probability of its occurrence.

calculation of which could be set out as in Table 4.2. However, to handle this kind of problem systematically it is helpful to refer to a tree diagram like that of Fig. 4.2. This diagram shows that there are four possibilities (four branches) for the outcomes of number of lambs born per ewe. For the outcome of no lambs born there is only 1 possible revenue outcome, namely zero, whereas for three lambs there are 8 possible revenue outcomes, not all different, depending upon the prices obtained for each of the 3 lambs. The length and direction of the lines have no significance but the branching enables us to plot all the possible ways in which all possible final outcomes might be achieved. The probabilities, conditional or otherwise, are marked alongside each branch and these may be used in either of two ways: to calculate the joint probabilities for each path—by multiplying together all the probabilities along the path—and thereby to arrive at the expected revenue in a similar manner to that used in Table 4.2; or to work backwards from the final outcomes calculating the expected money value at each junction, or node, of the tree. For instance, let us concentrate on the branch representing a single lamb, with the two final outcomes of either £8 revenue or £10 revenue. Working back from these final outcomes and 'averaging out' we obtain an EMV for a single lamb of

$$(0{\cdot}4 \times £8) + (0{\cdot}6 \times £10) = £9.2.$$

The whole exercise is completed in Fig. 4.3, which shows the over-all expected revenue per ewe to be £14.176.

In order to determine the expected price per lamb we must now take account of the probabilities of an individual lamb being a singleton, one of a twin, or one of a triplet. As we have seen the total number of lambs, born of 200 ewes, was 314 made up of 100 singles, 80 pairs of twins (160 lambs), and 18 sets of triplets (54 lambs). Hence the probabilities (relative frequencies) of a lamb being a singleton, one of a twin or one of a triplet are

$$\frac{100}{314} = 0{\cdot}318; \frac{160}{314} = 0{\cdot}51; \text{ and } \frac{54}{314} = 0{\cdot}172$$

respectively. These probabilities together with the conditional probabilities of receiving £8 or £10 per lamb are set out as a tree diagram in Fig. 4.4. The lower tree shows the derivation of expected values.

The upper branch of Fig. 4.4 shows that, if all lambs were singles, the expected price would be £9.2, as we have already calculated. Now from the centre branch we can calculate that, if all lambs were twins, the expected price per lamb would be

$$(0{\cdot}5 \times £8) + (0{\cdot}5 \times £10) = £9.$$

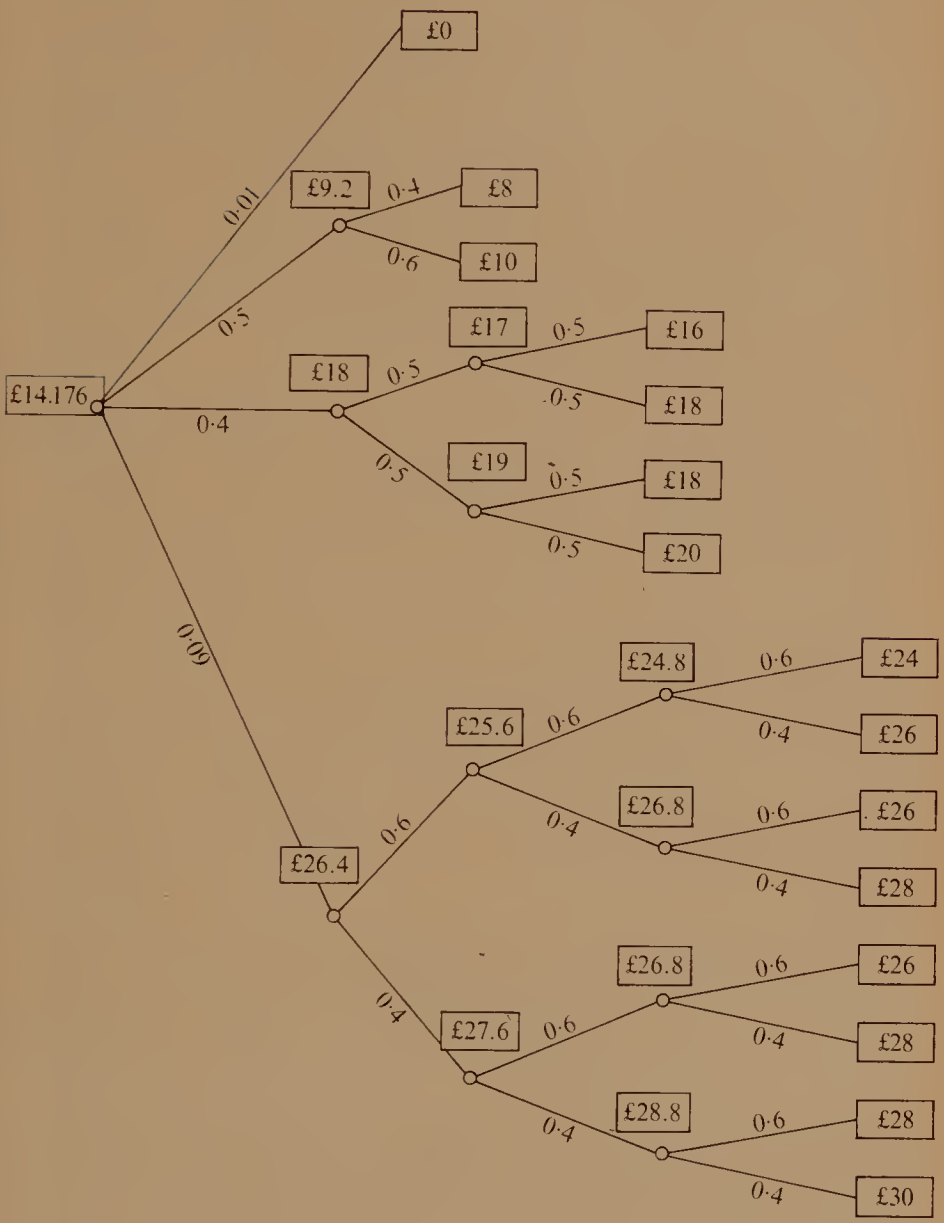


FIG. 4.3. Expected revenue per ewe. This is simply Fig. 4.2 with the corresponding expected monetary values entered at each node. These are derived by 'averaging out', i.e. by multiplying revenues at the end of branches to the right of the node by their corresponding probabilities and summing the results.

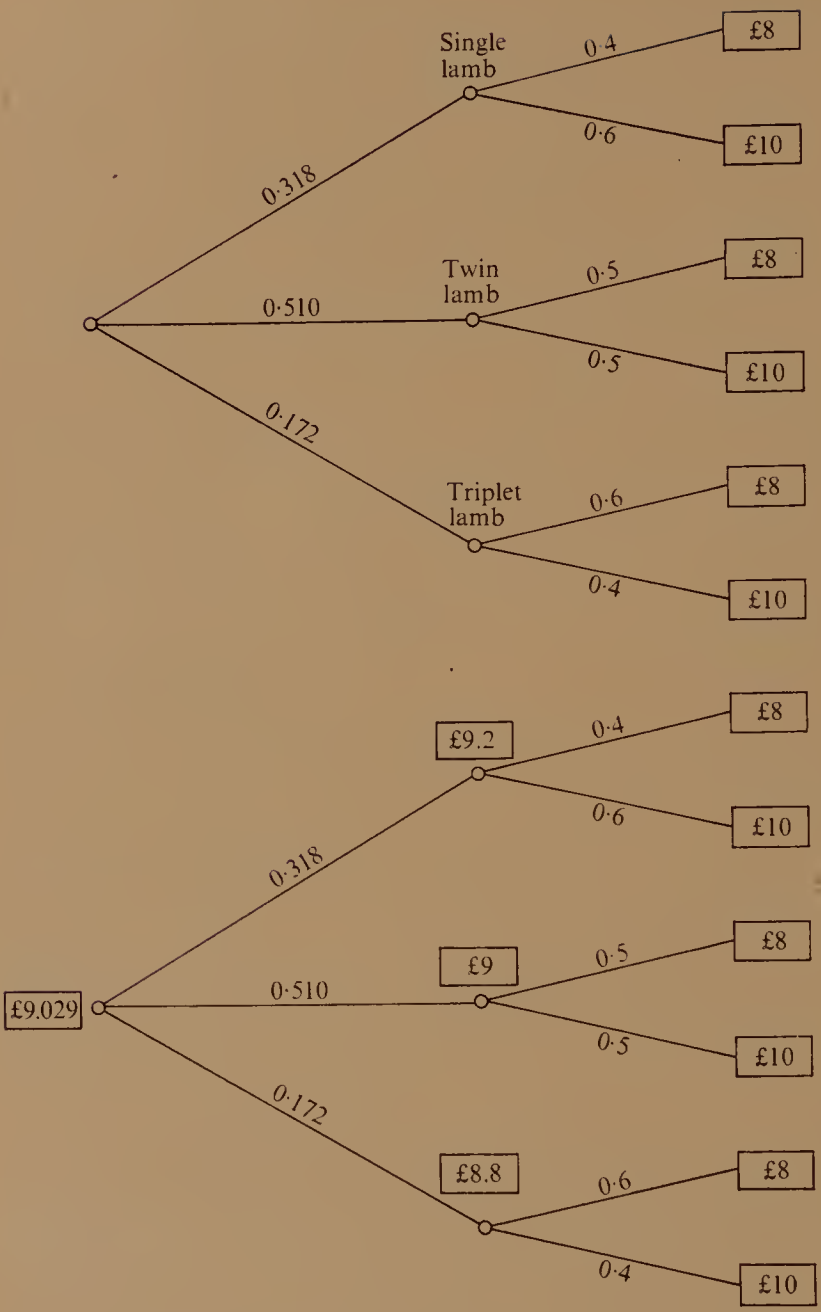


FIG. 4.4. Expected price per lamb. This tree diagram is used to determine the expected price per lamb rather than the expected revenue per ewe.

Finally from the lower branch we find that, if all lambs were triplets, the expected price would be only

$$(0{\cdot}6 \times £8) + (0{\cdot}4 \times £10) = £8.8.$$

Thus the expected price per lamb depends upon the size of litter the lamb comes from. For the flock in our example conditional expected price per lamb, given the expected yield of lambs is

$$(0{\cdot}318 \times £9.2) + (0{\cdot}51 \times £9) + (0{\cdot}172 \times £8.8) = £9.029.$$

The expected revenue per ewe is,

$$1{\cdot}57 \times £9.029 = £14.176$$

as we found before.

In fact, strictly speaking, all probabilities should be expressed as conditional probabilities, and all expectations should be expressed as conditional expectations. The expected number of lambs per ewe is conditional upon the breed of sheep, the physical environment, and the system of management. In fact the very concept of a production function is based on the assumption that the expected level of output is dependent upon the level of inputs. Expected prices are likely to be dependent upon not only the quality of the product but also time since; as we have seen, prices vary over time. Thus estimates of EMVs based on average yields and prices obtained in the past may prove unreliable in a changing environment.

The fallacy of averages

Some possible errors which may arise if we manipulate average yields and average prices as though they were actual yields and prices may be illustrated by means of some simple examples.

In the first case let us assume that expectations are based on the results obtained over a series of years in the past. In favourable seasons, occurring 3 years in 10, the lamb crop is 200 per cent or an average of 2 lambs per ewe, whereas in unfavourable seasons, occurring 7 years in 10, the lamb crop is only 100 per cent, or an average of 1 lamb per ewe. Thus over a period of years the expected number of lambs per ewe is

$$(0{\cdot}3 \times 2) + (0{\cdot}7 \times 1) = 1{\cdot}3.$$

The average price in favourable years is £8 per lamb but in unfavourable years, due to the shortage of lambs, the price rises to £10. One might then be tempted to assume that since a price of £8 is obtained in 3 years

out of 10 and a price of £10 is obtained in 7 years in 10, the expected price is

$$(0{\cdot}3 \times £8) + (0{\cdot}7 \times £10) = £9.4 \text{ per lamb.}$$

However, the expected revenue per ewe is

$$(0{\cdot}3 \times 2 \times £8) + (0{\cdot}7 \times £10) = £11.8,$$

which is not equal to

$$1{\cdot}3 \times £9.4 = £12.2.$$

The error arises because the true expected price per lamb is conditional upon the number of lambs born. Since, when the price is £8, twice as many lambs are born as when the price is £10, the price of £8 should be weighted twice as heavily in arriving at the expected price per lamb; thus

$$2\left(\frac{0{\cdot}3}{1{\cdot}3} \times £8\right) + \left(\frac{0{\cdot}7}{1{\cdot}3} \times £10\right) = £9.1.$$

The second example shows how in a dynamic planning situation, where decisions are made sequentially, expectations may differ from the average values. For this example we assume that the prices per lamb of £8 and £10 are equiprobable, i.e. they both have the probability 0·5. However, the farmer now has a choice of marketing dates. If he is not satisfied with the price which actually prevails when the lambs are ready for market, he can delay selling them until a month later. The probabilities of obtaining £8 or £10 at the second date are independent of the price prevailing at the first date and remain constant at 0·5. For simplicity we assume that the sale of lambs cannot be delayed beyond the second market date but that up to then there are no costs involved in delaying sales. In this case the expected price per lamb is *not* $(0{\cdot}5 \times £8) + (0{\cdot}5 \times £10) = £9$ because the decision maker has two opportunities to try for the higher price.

The problem may be worked out using the tree diagram in Fig. 4.5(a). This is now a *decision tree*, since it incorporates decision forks, each one marked with a small square; chance forks are marked with circles. Apart from the decision forks representing the choice of whether to sell at the first market date, another decision is incorporated at the outset, namely whether to give up fat lamb production and thereby save a cost per lamb of £9.2.

In drawing a decision tree it is important that the decision and chance forks are represented in the sequence in which they would occur in

practice. For instance, in this example, a different answer would be obtained if the farmer had to decide whether to sell on the first date *before* knowing the price outcome on that date.

The solution which yields the maximum EMV is determined in the lower diagram. Fig. 4.5(b). The method of solution is sometimes referred to as 'averaging out and folding back', or 'backwards induction'.

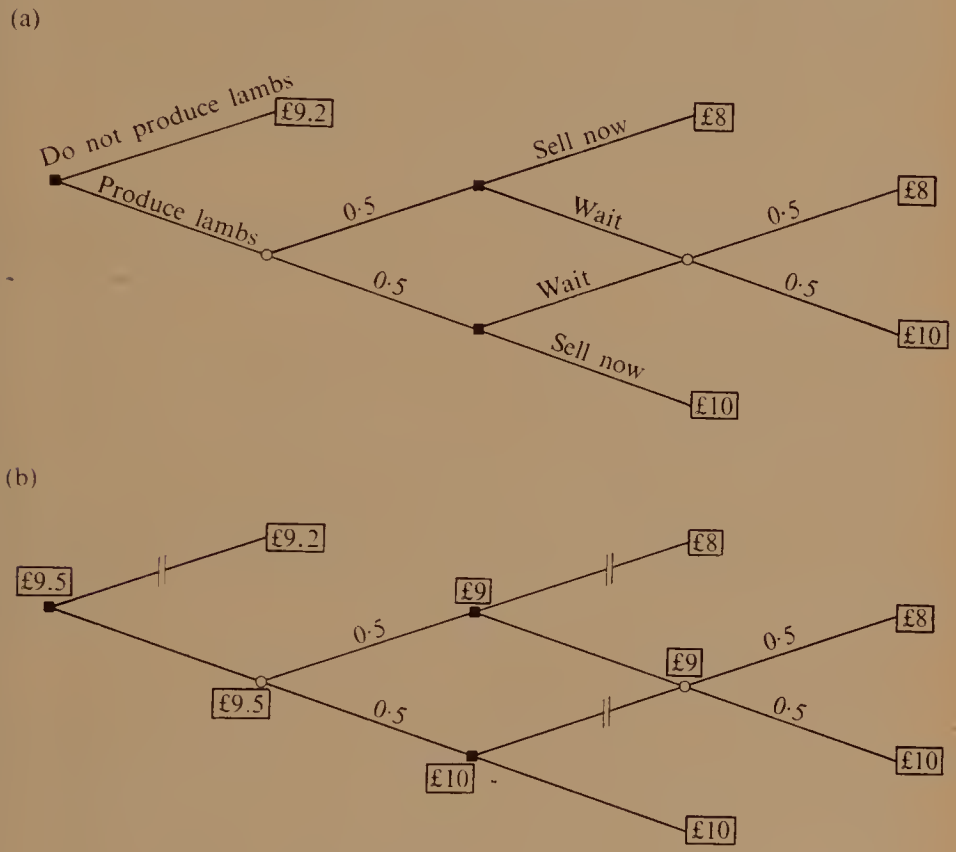


FIG. 4.5. Decision tree for sequential decisions. The tree now incorporates decision nodes (marked by squares) as well as outcome nodes (circles). At each decision node a conscious choice is made as to which of the branches to the right of it, will be eliminated (edited or pruned). Thus the EMV in this case is determined in part by chance outcomes and in part by conscious decisions.

Starting from the final outcomes on the right of the diagram, we work backwards, calculating the EMV at each chance fork. This is the process of averaging out. Thus the EMV for the second market date is £9, as already calculated. Now, at each decision fork a choice is made according to the EMVs of the alternative branches. The branch with the higher EMV is chosen and the other branch is crossed through. This is the

process of folding back. For example, if the price at the first market date turns out to be £8, the farmer has a choice of selling for £8 or delaying with an EMV of £9. To maximize EMV he should wait, so the branch representing 'sell now at £8' is crossed through. However, if the price on the first date turns out to be £10 he should sell because this is greater than an EMV of £9. If we ignore the branches which are crossed through, we find that, before the first market date, the farmer anticipates a price of £10 with a probability of 0·5, or an EMV of £9 with the same probability. The expected value of these two outcomes is therefore

$$(0{\cdot}5 \times £10) + (0{\cdot}5 \times £9) = £9.5.$$

Since this is greater than the alternative of saving £9.2 by not producing fat lambs, the decision to produce is taken. However, the important fact to note is that the expected price is £9.5 and not £9 as one might at first assume.

The final examples show how fixed constraints may prevent the farmer from taking advantage of better-than-average yields, so the expected *usable* yield is less than the expected *total* yield. This situation would arise in our lamb-production example if ewes were incapable of rearing triplets or of fostering orphan lambs so that it was necessary to give away 1 lamb from each set of triplets as a pet. Then although the average number of lambs *born* per ewe, that is the expected total yield, would still be

$$(0{\cdot}01 \times 0) + (0{\cdot}5 \times 1) + (0{\cdot}4 \times 2) + (0{\cdot}09 \times 3) = 1{\cdot}57,$$

the average number raised per ewe, or the expected usable yield, would be reduced to

$$(0{\cdot}01 \times 0) + (0{\cdot}5 \times 1) + (0{\cdot}49 \times 2) = 1{\cdot}48.$$

This kind of situation may cause errors in linear programming, as may be demonstrated by the simple problem set out in Table 4.3. There are just two activities, grain and root production, and two constraints, land and root-harvest labour. Prices and grain yields are expected to remain constant but root yields vary from 5 tonnes per hectare in unfavourable (poor) years to 15 tonnes per hectare in favourable (good) years. Good years and poor years occur with equal probability. The expected root yield per hectare is therefore

$$(0{\cdot}5 \times 5) + (0{\cdot}5 \times 15) = 10 \text{ tonnes},$$

so that the expected land requirement per tonne is 1/10 hectares.

It is arguable that the expected land requirement per tonne should be calculated as $(0{\cdot}5 \times (1/5)) + ((0{\cdot}5 \times (1/15)) = 2/15$ as noted in Table 4.3. Both alternatives seem to be justifiable but the following argument would apply in either case.

TABLE 4.3
Linear programming under risk

(a) The problem

	Constraint level	Grain production (per bag)	Root production (per tonne)		
			Good years	Poor years	Expected values
Gross margin per unit (£)		2	7	7	7
Land (hectares)	50	1/10	1/15	1/5	1/10†
Root harvest labour (units)	10	0	1/30	1/30	1/30

† The expected land requirement per tonne is estimated from the expected yield per hectare:

$$(0{\cdot}5 \times 15) + (0{\cdot}5 \times 5) = 10 \text{ tonnes},$$

and not directly from the land requirements per tonne in good and poor years.

(b) The solutions

	Gross margin in 'good' year (£)	Gross margin in 'poor' year (£)	Expected value (£)
(i) 'Average expectations' plan (20 hectares grain; 30 hectares roots)	2500	1450	1975
(ii) Optimistic (good year) plan (30 hectares grain; 20 hectares roots)	2700	1300	2000
(iii) Pessimistic (poor year) plan (zero grain; 50 hectares roots)	2100	1750	1925

Using the expected requirement of 1/10 hectare per tonne of roots, the maximizing solution is found to be 20 hectares of grain and 30 hectares of roots, as shown at point A in Fig. 4.6, with a total gross margin of

$$(200 \times £2) + (300 \times £7) = £2500.$$

This solution is *wrong*.

First let us consider the result in poor years when 30 hectares of

roots will yield only $30 \times 5 = 150$ tonnes. This result, is shown in Fig. 4.6 at A′. The total gross margin is then

$$(200 \times £2) + (150 \times £7) = £1450.$$

Now consider total gross margin in good years when 30 hectares will yield 450 tonnes. However, the amount of labour available limits the quantity of roots that can be harvested to 300 tonnes, so the result in good years is also represented at point A.

Hence the total gross margin in good years is

$$(200 \times £2) + (300 \times £7) = £2500.$$

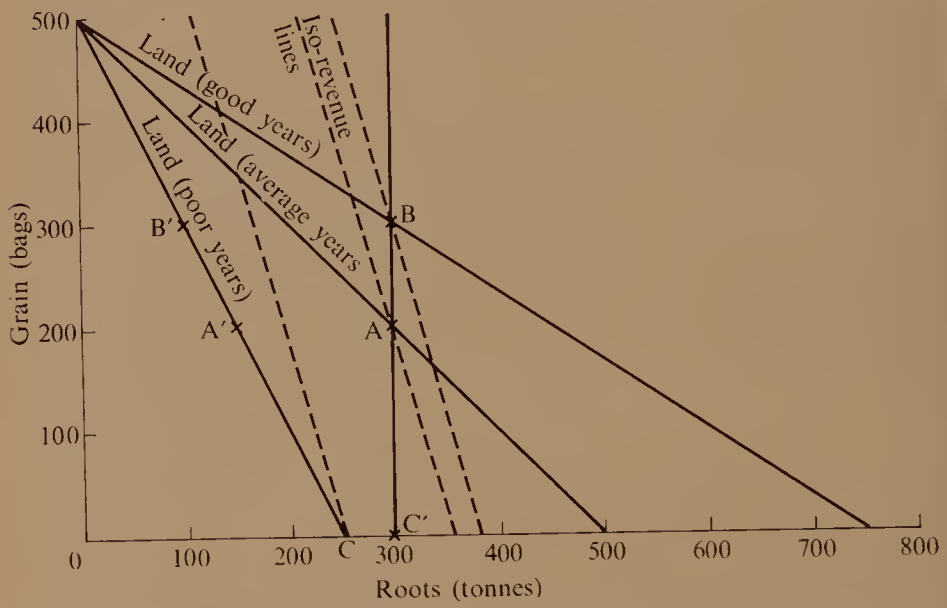


FIG. 4.6. Linear programming optima under risk. The diagram shows the production-possibility boundaries for favourable (good) years and unfavourable (poor) years, and for the average of these. The iso-revenue lines are also shown. Whereas on average the optimal solution is at point A, for good years it is at point B, and for poor years at point C.

The 'expected' total gross margin over good years and bad will then be

$$(0{\cdot}5 \times £1450) + (0{\cdot}5 \times £2500) = £1975,$$

not £2500 as estimated previously.

However, the more serious aspect is that this solution does not maximize EMV. It is not the real optimum in good years, poor years, or on average.

To illustrate, we will use linear programming to establish the profit-maximizing solutions for each alternative root yield. First, let us take the optimistic view and plan using the root yields for good years. The optimal solution then consists of 30 hectares of grain and 20 hectares of roots, as shown at point B in Fig. 4.6. The 20 hectares of roots yield 300 tonnes in good years so the total gross margin is

$$(300 \times £2) + (300 \times £7) = £2700.$$

Now, in poor years 20 hectares of roots will yield 100 tonnes as represented at point B′ in Fig. 4.6 so the total gross margin is

$$(300 \times £2) + (100 \times £7) = £1300.$$

The EMV for this plan (plan ii) is therefore

$$(0{\cdot}5 \times £2700) + (0{\cdot}5 \times £1300) = £2000,$$

which is greater than the £1975 expected from the original plan. The results for plans based on expected, good, and poor yields are given in part (b) of Table 4.3. The optimum plan when the pessimistic view is taken, and the low root yield is assumed, occurs at point C in Fig. 4.6. In good years the result will be as shown at C′. Clearly then, if the objective is to maximize expected monetary value, plan ii based on optimistic estimates gives a better result than plan i based on expected values.

Risk aversion

Despite the possible fallacy of using averages, the methods described above are relatively straightforward. There is a single value to be maximized, namely the EMV, and if the probability distribution of money gains and losses is known, it is uniquely determined. However, we must now face up to the fact that most people have an aversion to risk; if faced with a choice between the certainty of receiving £50 or a 50:50 chance of receiving £100 or nothing they would choose the former even though the EMV is the same in each case. This is now clearly a subjective matter; a few individuals may prefer to gamble on the risky option. Furthermore, of those who prefer the certain outcome some would change their minds if it was reduced to £40, they would then choose the 50:50 chance of receiving £100 or nothing, but others would not do so until the certain payment was reduced to £30 or less. No general prescription of the '*right*' decision is possible.

In theory the avoidance of risk has a cost. In so far as profits are likely to be maximized in the long run by always choosing the alternative

having the highest EMV, some profits will be forgone if any other criterion is used. However, many of the decisions farmers have to take are not repeated many times—indeed, some are unique—so there is no opportunity for gains to cancel out losses. In such circumstances it is perfectly rational to avoid risk at the expense of some profit, especially in those situations where a few large losses could put the farmer out of business. Indeed it is usually considered prudent to pay for insurance of major possessions against loss, yet the cost of the annual premiums must exceed the EMV of the benefits, otherwise the insurance companies would be unable to meet *their* costs.

The need for taking risk into account is particularly marked where the probability distribution of money gains and losses is skewed, which means that it is not symmetrical about the expected value. In such a case the most probable outcome, called the mode, or modal outcome, differs from the expected value. For instance, from Table 4.1 it is clear that the distribution of numbers of lambs per ewe is skewed so the most probable outcome is for an ewe to bear 1 lamb although the expected number of lambs per ewe is 1·57. Thus a few high values with a low probability will pull the expected value up above the most probable result. For a simple but extreme example consider an activity with a 0·9 probability of yielding £100 return and a 0·1 probability of yielding £1000. The expected return is then £190. Many farmers would prefer to base their decision on the assumption of £100 return although logically the possibility of the very high return should also be taken into account.

The riskiness of a particular activity is dependent upon the variability of outcomes. This variability may be estimated from the probability distribution of outcomes, the measure most often used being the variance. Variance is calculated by taking the difference between each outcome and the mean, giving what is known as the deviation from the mean, squaring, then multiplying by the probability and summing for all possible outcomes.[2] Thus from the data given in Table 4.1 the variance

[2] It has been found that this method slightly underestimates the true variance, if based on a limited number of observations. The bias may be corrected by multiplying by the ratio of the number of observations to one less than the number of observations. Thus Table 4.1 is based on 200 observations of individual ewes so the above estimated variance of 0·4451 should be multiplied by 200/199 to give

$$0{\cdot}4451 \times \frac{200}{199} = 0{\cdot}4474.$$

However this correction becomes really important only when working with small samples.

of the number of lambs per ewe is calculated to be

$$0{\cdot}01(0 - 1{\cdot}57)^2 + 0{\cdot}5(1 - 1{\cdot}57)^2 + 0{\cdot}4(2 - 1{\cdot}57)^2 + 0{\cdot}09(3 - 1{\cdot}57)^2$$
$$= 0{\cdot}01(-1{\cdot}57)^2 + 0{\cdot}5(-0{\cdot}57)^2 + 0{\cdot}4(0{\cdot}43)^2 + 0{\cdot}09(1{\cdot}43)^2 = 0{\cdot}4451.$$

Sometimes the 'standard deviation', which is the positive square root of the variance, is found to be more useful than the variance since it is in the same units as the original variable rather than in squared units. In this instance the standard deviation is $\sqrt{0{\cdot}4451} = 0{\cdot}67$ lambs.

Now we have two different measures for comparing alternative actions or plans, namely (1) the expected value and (2) the variance of the monetary outcomes. Since in serious business matters most people including most farmers, are risk averse it is reasonable to assume that where there are several alternative policies all yielding the same EMV, the one with the lowest variance will be preferred.[3] Decision makers are concerned to minimize variance as well as to maximize EMV.

Diversification and risk

Diversification is often recommended as a precaution against risk on the grounds that income variability (variance) is likely to be less if the farmer avoids having all his eggs in one basket. We have seen in earlier chapters that diversification may be profitable, i.e. it may be a means of increasing EMV where there are supplementary or complementary relationships between enterprises. However, in this section we are concerned with the separate effect of diversification on income variance. In many cases it will have the effect of reducing variance but sometimes a single reliable activity may yield less variable returns than any combination of activities.

For simplicity let us assume just two alternative activities, namely grain production and root production, each with only two alternative financial outcomes, depending upon whether the season is wet or dry. Let us also assume that wet and dry seasons occur with equal probability, so that the probability for each outcome is $\frac{1}{2}$, or 0·5. Note, however, that the analysis may be generalized to situations where there are many alternate activities and the probability distributions of outcomes differ from the one we assume here.

[3] Those who enjoy gambling must in effect prefer risk but this preference generally holds only when relatively small sums of money are involved; that is sums which are small in relation to total annual income. Most of us prefer a secure main source of income. Note that decision makers may be concerned with other characteristics of the distribution of outcomes: the skewedness, for instance. However, the EMV and variance are likely to be the most important characteristics.

First consider the figures given in Table 4.4. The gross margin per hectare of grain is £20 in wet years and £30 in dry years so the EMV is

$$(\tfrac{1}{2} \times £20) + (\tfrac{1}{2} \times £30) = £25$$

and the variance is

$$(\tfrac{1}{2} \times 5^2) + (\tfrac{1}{2} \times 5^2) = 25.$$

For roots the corresponding gross margin are £30 and £10 so the EMV is

$$(\tfrac{1}{2} \times £30) + (\tfrac{1}{2} \times £10) = £20$$

and the variance is

$$(\tfrac{1}{2} \times 10^2) + (\tfrac{1}{2} \times 10^2) = 100.$$

TABLE 4.4

Risk reduction by diversification; inverse relationship between activities

	Wet years	Dry years		
Probability of occurrence	½	½		
	Gross margin per hectare		Expected value	Variance
	(£)	(£)	(£)	
Grain	20	30	25	25
Roots	30	10	20	100
½ grain: ½ roots	25	20	22½	6¼
⅔ grain: ⅓ roots	23⅓	23⅓	23⅓	0

The third row of the table shows the results of diversifying by using ½ the available land for grain and ½ for roots. The average gross margin per hectare in wet years is then the average of £20 and £30, namely £25, and in dry years the average of £30 and £10, which is £20. These two gross margins of £25 and £20 have an EMV of £22.50 and a variance of 6¼ which is considerably lower than the variance for either of the crops grown alone. In fact, if grain and roots are combined in the proportions ⅔ to ⅓, the average gross margin is £23.33 per hectare in both wet and dry years so that the variance is reduced to zero.

Now turn to Table 4.5. It will be noted that, whereas in Table 4.4 low returns from grain are associated with high returns from roots and

vice versa, in Table 4.5 the returns for the two crops are shown moving together. The EMV and the variance for each crop is the same as before but now the variance for any combination of the two is greater than the variance for grain alone. Thus it appears from these simple examples that variability of income is reduced by diversification only if the returns from the two activities vary in opposite directions. In fact this is not the whole story but first we need a measure of the relationship between two variables.

Such a measure is the covariance, which is calculated by multiplying the product of the deviations from the mean for each outcome by the

TABLE 4.5

No risk reduction by diversification; direct relationship between activities

	Wet years	Dry years		
Probability of occurrence	½	½		
	Gross margin per hectare		Expected value	Variance
	(£)	(£)	(£)	
Grain	20	30	25	25
Roots	10	30	20	100
½ grain : ½ roots	15	30	22½	56¼

probability and summing for all possible outcomes. Thus the covariance of the gross margins for grain and roots given in Table 4.4 is calculated to be

$$\tfrac{1}{2}(20 - 25)(30 - 20) + \tfrac{1}{2}(30 - 25)(10 - 20) = -50.$$

Note that because $(20 - 25)$ is a negative deviation and $(30 - 20)$ is a positive one their product is negative. Similarly the product of $(30 - 25)$ and $(10 - 20)$ is negative so that total covariance is negative. However, for the data given in Table 4.5 the covariance is

$$\tfrac{1}{2}(20 - 25)(10 - 20) + \tfrac{1}{2}(30 - 25)(30 - 20) = +50.$$

In this case the negative deviations $(20 - 25)$ and $(10 - 20)$ are multiplied together so that their product is positive and so too is the covariance. Thus a negative covariance implies that the variables are inversely

related, in which case we say there is a negative correlation.[4] A positive covariance implies that the variables are directly related: there is a positive correlation. A covariance of zero means that there is no relationship between the variables; they are not correlated.

The general rule is that income variability can be reduced only if the covariance is smaller than the variance of each single activity (for proof see the mathematical appendix to this chapter). Thus when, as was the case for the example of Table 4.4, the covariance is negative it must be smaller than the variance of either single activity, since variance

TABLE 4.6

Covariance positive but less than variance

	Wet years	Intermediate years	Dry years			
Probability	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$			
	Gross margin per hectare			Expected value	Variance	Covariance
	(£)	(£)	(£)	(£)		
Grain	20	25	30	25	$16\frac{2}{3}$	$+13\frac{1}{3}$
Roots	20	30	28	26	$18\frac{2}{3}$	
$\frac{1}{2}$ grains: $\frac{1}{2}$ roots	20	$27\frac{1}{2}$	29	$25\frac{1}{2}$	$15\frac{1}{2}$	

is always positive. Clearly, in that case variation will be reduced by diversifying. However, in the second example, the covariance of + 50 is greater than the variance for grain alone which is 25. Hence variance is minimized by producing grain alone. It is quite impossible to reduce variance by diversification in this case.

Alternative situations are illustrated by the hypothetical data given in Tables 4.6, 4.7, and 4.8. In each of these examples we assume *three* equiprobable outcomes which means that, unlike the case when there are only two outcomes, the relationship between the returns from the two crops need not be exact. The data given in Table 4.6 illustrates

[4] The correlation coefficient is defined as the covariance divided by the geometric mean of the two variances; the geometric mean being the positive square root of the product. Hence the correlation coefficient must be positive if the covariance is positive and negative if the covariance is negative. The correlation coefficient is used rather than the variance to measure strength of association, since it is independent of the units of measurement of the two variables.

a situation where the covariance of the returns for the two crops is positive but less than the variance for either single crop. As a result the variance for a half-and-half combination of grain and roots is less than that for either crop grown alone.

Table 4.7 illustrates a situation where covariance is equal to the variance of the less variable activity grain. In this case the variability of the gross margin per hectare of a combination of grain and roots is greater than the variability for grain alone. The reader may check that any other combination of the two crops would have a variance greater than $16\frac{2}{3}$.

TABLE 4.7

Covariance positive but equal to variance

	Wet years	Intermediate years	Dry years			
Probability	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$			
	Gross margin per hectare			Expected value	Variance	Covariance
	(£)	(£)	(£)	(£)		
Grain	20	25	30	25	$16\frac{2}{3}$	$+16\frac{2}{3}$
Roots	20	30	30	$26\frac{2}{3}$	$22\frac{2}{9}$	
$\frac{1}{2}$ grain: $\frac{1}{2}$ roots	20	$27\frac{1}{2}$	30	$25\frac{5}{6}$	$18\frac{1}{3}$	

Even if the covariance is less than the variance of any single activity, variability may not be reduced by diversification if the activities are not combined in the right proportions. This is particularly important if the second activity has a much larger variance than the first. Thus Table 4.8 illustrates a situation where the covariance is less than the variance of gross margins for grain but where that for roots is much larger. A half-and-half combination of grain and roots then *increases* the variance per hectare. However, the last line of the table shows that it is just possible to find some combination of the two activities with a lower variance than grain alone. Thus it is still true to say that income variance *can* be reduced by diversification if the covariance is less than the variance for each activity alone, but clearly this is not to say that it will.

Thus the following practical conclusions may be drawn regarding the effects of diversification in reducing income variation.

1. If there is a negative correlation in returns from two different activities, or even if there is no relationship at all, diversification may reduce income variation. Unfortunately, positive correlations are far more likely in practice. In a poor growing season all crop yields are likely to be below average, while in a good season all crop yields will probably be above average. Similarly, product prices all tend to move together in the same direction.

However, the returns from crops and livestock enterprises may not

TABLE 4.8

Second activity much more variable than the first

	Wet years	Inter-mediate years	Dry years			
Probability	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$			
	Gross margin per hectare			Expected value	Variance	Covariance
	(£)	(£)	(£)	(£)		
Grain	20	25	30	25	$16\frac{2}{3}$	$+11\frac{2}{3}$
Roots	10	30	17	19	$68\frac{2}{3}$	
$\frac{1}{2}$ grain: $\frac{1}{2}$ roots	15	$27\frac{1}{2}$	$23\frac{1}{2}$	22	$27\frac{1}{6}$	
$\frac{9}{10}$ grain: $\frac{1}{10}$ roots	19	$25\frac{1}{2}$	$28\frac{7}{10}$	$24\frac{2}{5}$	$16\frac{2}{7}$	

be related, so that diversification into mixed farming may reduce income variance.

2. If there are two or more activities with similar variances then it is quite likely that the covariances, although positive, will be less than the variances. Again, in this situation income variation may be reduced by diversification.

3. If on the other hand there is a single reliable enterprise, the returns from all other activities being considerably more variable, then it is likely that covariance will exceed the variance of the reliable enterprise. Diversification will then *increase* income variability. Even if the covariance is less than the variance for the reliable enterprise, diversification may still increase income variability as shown in the last example.

In this discussion we have considered only relatively simple situations for which the variance per hectare of an enterprise or a combination of enterprises is assumed to be constant regardless of the scale of production. Where variance per unit of production varies with scale and where there are constraints on output, the situation is more complex, and the conclusions about the effects of diversification may not apply. Thus results given in Table 4.3 illustrate a situation where the least variable total revenue is obtained by specializing in the activity with the most variable revenue per hectare!

Clearly we cannot easily generalize about the effects of diversification; it may increase or decrease income variability. Similarly it may increase or decrease total EMV. Where diversification is beneficial both in increasing EMV and decreasing income variance, then presumably most farmers would choose to diversify. However, where both EMV and variance are increased (or decreased) together, the question of whether to diversify become a part of the broader problem of how to choose between alternatives given the twin objectives of EMV maximization and variance minimization.

Choices under risk

When there are large numbers of alternative policies under consideration it may be possible to construct a 'risk–earnings possibility curve', also referred to as an 'expectations–variance (E–V) efficiency frontier', like the hypothetical example illustrated in Fig. 4.7. Such a curve may be constructed by calculating the EMVs and variances for large numbers of alternative plans, selected in any reasonable way.[5] The results may then be plotted on a graph as shown, and a smooth curve drawn through the points representing the least variable plans for each EMV level. It is generally safe to assume that combinations of plans are feasible and that the frontier is continuous as shown. Alternatively, techniques are available for calculating the minimum variance plan for a given level of EMV.[6] The curve may then be constructed by finding the minimum variance plans for a series of different EMVs.

[5] Simulation techniques may be used for this purpose; see for instance Dillon, J. L. (1971). Interpreting systems simulation output for managerial decision making. In *Systems analysis in agricultural management* (ed. J. B. Dent and J. R. Anderson). Wiley, London.

[6] The technique of 'quadratic programming' may be used to find minimum variance plans, see for instance, McFarquhar, A. M. M. (1961). Rational decision making and risk in farm planning. *J. Agric. Econ.* **14**, 552.

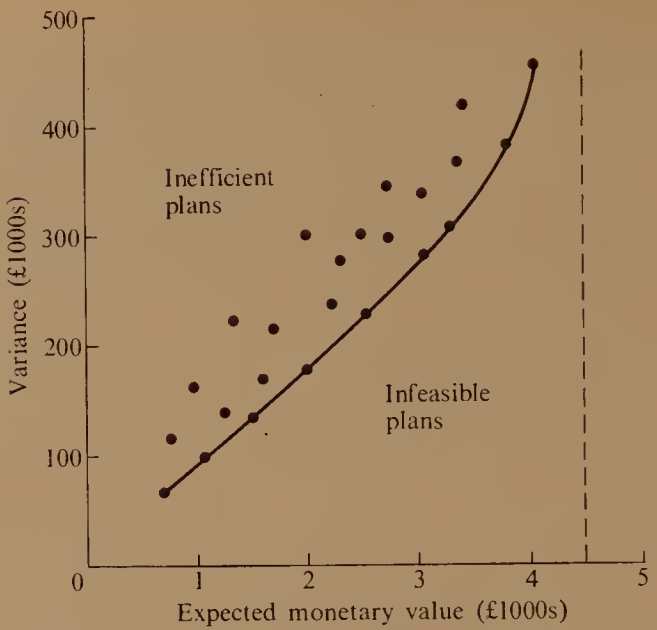


FIG. 4.7. The expectations–variance (E–V) efficiency frontier. Given the probability distribution of financial outcomes for each alternative plan of action, comparisons may be made on the basis of both means (EMVs) and variances. Each point marked on the diagram represents the EMV–variance combination for a specific plan. Those plans represented by points on the south-eastern boundary (the expectations–variance efficiency frontier) are efficient in the sense that variance is minimized for each EMV or equivalently that EMV is maximized for each variance level. The range of rational choice may be restricted to points lying on this frontier.

Plans represented by points above and to the left of the curve in Fig. 4.7 are 'inefficient' whereas results lying below and to the right are infeasible. The curve slopes upwards to the right, indicating that increased EMVs can only be achieved at the cost of increased variance. Furthermore the variance generally increases at an increasing rate as the maximum possible EMV is approached.

Now, it may be assumed that the best plan for an individual farmer will lie somewhere on this frontier, just as in the absence of risk the optimum combination of enterprises was assumed to lie on the production-possibility boundary. However, there is no variance: EMV price ratio, which may be used like the product: product price ratio to find the optimum combination. Indeed, since the extent of risk aversion varies from person to person, the best decision for one farmer may be far from the best for another. Thus we have a prescriptive (normative) theory that farmers ought to operate on the E–V efficiency frontier since such behaviour seems rational. However before we can prescribe specific plans as being the optimum for individual farmers

we need some means of measuring and predicting their attitudes to risk (positive theories).

There are two alternative theories as to how a decision maker may take into account two or more different objectives—such as EMV maximization *and* variance minimization—in comparing plans. One theory is that he has a subjective rate of trade-off, which means that he can assess how much expected income he is willing to forgo in order to achieve a given reduction in variance. If he can assess the precise amount he is willing to forgo in order to reduce the variance to zero, he can define a 'certainty equivalent' for the returns from each risky plan. Essentially a 'certainty equivalent' is the guaranteed, safe sum of money the decision maker would be just willing to exchange for the set of variable returns predicted for the plan. Different plans may then be compared in terms of their certainty equivalents.

An alternative theory is that the decision maker ranks his objectives in order of priority. He then makes his choice on the basis of the primary objective, but if there are several plans, all of which are equally attractive in terms of the primary objective, he chooses among these on the basis of his second objective.[7] If two or more plans are ranked equally according to both the first and the second objective then the choice is based on a third objective, and so on. This theory does not allow for the possibility of a trade-off between risk and expected gain. If the primary objective is to maximize EMV while variance minimization is the secondary objective, then so long as one plan yields a higher EMV than all others, that one will be chosen. However if two or more plans yield the same maximum EMV then the one with the smallest variance will be selected.

In practice it may be more appropriate to treat the primary objective as a constraint. Plans which do not satisfy the objective are then rejected as being infeasible. For instance, a farmer's first concern may be to ensure that his income does not fall below the minimum necessary for survival, even under the most adverse conditions. His primary objective is survival, or avoidance of ruin, rather than variance minimization. Thus his choices are limited to plans which provide for survival, and are made on the basis of a secondary objective, which may be EMV maximization. Indeed this approach may be extended to cases where there are more than two objectives by treating all but the least important as constraints, and the least important as the maximization of some

[7] This is known as a 'lexicographic ordering', meaning the approach used by dictionary makers. Thus in a dictionary words are arranged first in order of the first letter, then in order of the second letter, and so on.

quantity. However for the purposes of this discussion we consider only the case of EMV maximization subject to survival constraints.

Such problems are readily handled within the linear programming framework, provided that the necessary data can be acquired. In addition to the data needed for an ordinary profit maximizing programme, we need to know the year-to-year variation in returns from each activity and the basic survival needs of the individual farmer. A simple hypothetical example is set out in Table 4.9. The problem is

TABLE 4.9

Optimization subject to a survival constraint

	Constraint level	Grain production	Root production
Expected gross margin per unit (£)		25	20
Survival need wet years (£)	−1100	−20	−30
Survival need dry years (£)	−1100	−30	−10
Land (hectares)	50	1	1

			Expected gross margin
Optimum solution			
Grain production	40 hectares at	£25 =	1000
Root production	10 hectares at	£20 =	200
Surplus income over survival needs in dry years = £200			
Total expected gross margin			1200
Shadow prices			
Land	£35 per hectare on 50 hectares	=	1750
Survival need in wet years	−£0.50 per £1 on £1100	=	−550
Total expected gross margin			1200

presented here in terms of cash requirements for survival and variable cash returns from the crop activities, but it could equally well be analysed in physical terms, calories for instance.

The estimates of the year-to-year variation in crop returns are taken from Table 4.4. In addition we assume a minimum annual income, necessary for survival, of £1100 and a land constraint of 50 hectares. This problem is readily solved arithmetically by the methods described in the previous chapter but the graphical solution is shown in Fig. 4.8. Note that, since the annual income must be greater than or equal to the minimum necessary for survival, the feasible area must lie above the

survival constraint lines, as shown by the arrows. Land-use, on the other hand, must be less than or equal to the amount of land available so the feasible area lies below the land constraint line. As a result the feasible area is limited to the small triangle ABC. Clearly, in some parts of the world, the need to ensure survival seriously limits the range of choice of farming systems.

The optimum plan which maximizes EMV is found at point A, representing 40 hectares of grain and 10 hectares of roots, which yield a total EMV of £1200. Without the survival constraints the optimum plan

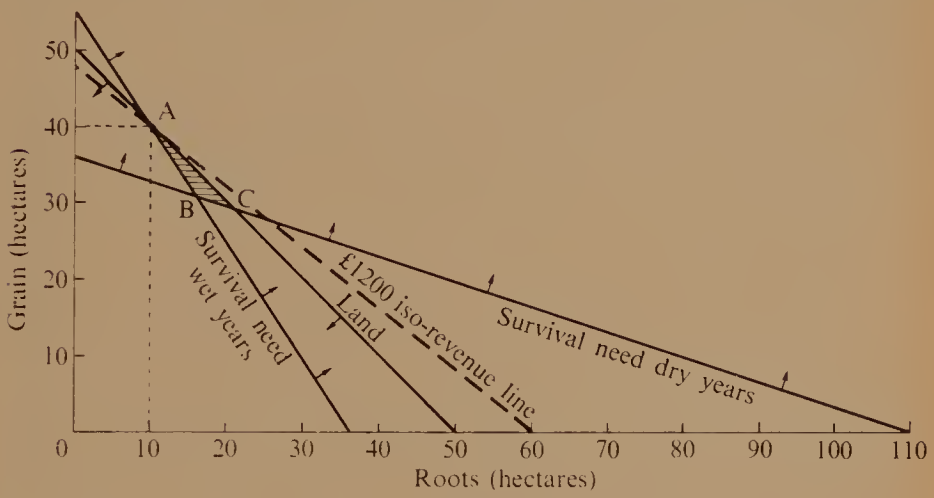


FIG. 4.8. Optimization subject to survival constraints. The land constraint line represents the *maximum* amount of grain and roots that can be grown. Hence feasible combinations are represented by points lying below and to the left of this line (see arrows). The survival constraint lines show the *minimum* amounts of the two crops which must be grown in order to satisfy the survival needs. Hence feasible combinations lie above and to the right of these lines. The feasible area is thus limited to the shaded triangle ABC. The highest possible iso-revenue (EMV) line touches the boundary at point A. Note that point B, representing a 2:1 combination of grain and roots, is a maximin point.

would be 50 hectares of grain yielding a total EMV of £1250 but providing an income of only £1000 in wet years. This approach is readily extended to real world problems where there are more alternative activities, more alternative climatic conditions, and more resource constraints.

Rather than determining a single optimum it may be more useful to explore the relationship between EMV and worst possible outcome when both are allowed to vary. This may be achieved, for instance, by repeated solution of a linear programme, like the one just described,

with a different survival requirement assumed for each solution. It is then possible to plot the curve relating EMV to worst possible outcome, like that in Fig. 4.9. Alternatively such a curve may be derived from the E–V frontier. Generally speaking the possibility of ruin cannot be eliminated entirely but it is often assumed that we may ignore the very worst possible outcomes with a small probability, say 0·05 (or 1 chance in 20), of occurrence. Then estimates of the EMV and variance of an efficient plan may be used to calculate the lower limit below which only 0·05 of all outcomes will fall. If the shape of the probability distribution

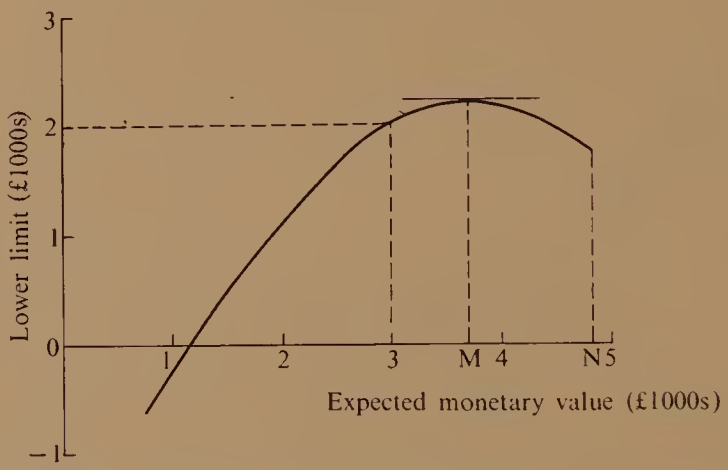


FIG. 4.9. Expected monetary values and associated lower limits. These may be used, rather than EMV and variance, to characterize plans with risky outcomes. It is generally found that up to a point, the lower limit (worst possible outcome) increases with increasing EMV but eventually the lower limit curve passes a maximum (maximin) point and then gradually falls. In this case the area of choice between increasing EMV and increasing risk lies to the right of the maximin point.

of outcomes is known, the lower limit can be established more accurately.

When the lower limits for a set of minimum-variance plans are plotted against their EMVs the graph is generally similar to the hypothetical example illustrated in Fig. 4.9. Thus for a plan with an EMV of £3000 the long-run average income is £3000 and the 'lower limit' is £2000. Initially, until EMV reaches OM, plans with higher EMVs also have higher minimum incomes, although the lower limit does not increase as fast as EMV because variance (i.e. the spread of incomes) is also increasing. In some situations the lower limit goes on increasing, so that the plan which maximizes EMV also maximizes the minimum income level. In other cases there is a point of inflection, as shown in the figure, beyond which the effect of increasing variance outweighs the effect of increasing

EMV so that the lower limit falls. The point of inflection then represents the maximum level of the minimum incomes, in this case about £2200. The policy which produces this result is therefore known as a 'maximin policy' or 'maximin plan'.

Now, if the farmer's two main objectives are first to avoid ruin and secondly to maximize EMV, it is irrational for him to choose any plan with an EMV of less than OM, since the maximin policy is preferable on both counts; it yields a higher EMV and higher minimum income. Thus the most cautious policy it is rational to pursue is the maximin one. Those farmers who are willing to take risks and accept a smaller minimum income than £2200 in this example, can adopt plans with EMVs greater than OM. In short, we now suggest that the rational area of choice is limited to those plans with EMVs between OM and ON, the latter being the maximum feasible EMV. The final decision as to which plan to adopt within this range is still a matter for individual choice depending upon the farmer's survival needs.

Expected utility maximization

Risk aversion may be viewed as a reflection of the diminishing marginal utility of money. From this viewpoint it is argued that a guaranteed sum of £50 is preferred to a 50:50 chance of £100 or nothing because the satisfaction or utility to be gained from an extra £100 is less than twice that to be gained from an extra £50. Put more formally, the utility of the second £50 is less than that of the first £50; the marginal utility of money diminishes. It follows then that for those with a preference for risk, who enjoy gambling, the marginal utility of money increases as income increases.

Now clearly the utility of a given sum of money, and the rate at which the marginal utilities of additional sums of money diminish, varies from one person to another, depending upon his current level of income and savings, his personality and many other factors. However if an individual farmer, the 'decision maker', could tell us the utility he would gain from different sums of money we could then define the optimum or best choice of plan for him to pursue. Naturally we could not generalize from this; the optimum plans for other decision makers would be different.

In fact it is unlikely that our decision maker would be able to tell us the utility he would derive from a specific sum, say £100. For a start there is no scale of units of utility against which to evaluate £100. However, if he is able to evaluate a series of risky alternatives in money terms, and if he is reasonably consistent in his choices, it is possible to

construct a scale for measuring his relative utility of different sums of money.

The basis of the method is to present the decision maker with a series of 'choices' of the form; 'what sum of money would you be just willing to exchange for a 50:50 chance of winning £5000 or losing £1000?'[8] In other words, he is asked to define his certainty equivalent in money terms for the gamble. Suppose, in this example, the decision maker says his certainty equivalent is + £400. This may be interpreted to mean that he is indifferent as to whether he receives an assured sum of £400 or the opportunity to gamble for the equiprobable outcomes of £5000 gain or £1000 loss.

The sums of money involved in the initial gamble, in this case + £5000 and −£1000, are intended to cover the whole range of possible monetary outcomes the decision maker is likely to meet with in practice. All other monetary outcomes will then be evaluated in relation to these two sums. For this purpose we ascribe to them numbers, which may be referred to as the numbers of units of utility (utils) represented by each monetary outcome, though we are in no way measuring actual utility. The choice of these numbers is quite arbitrary except that the utility of £5000 gain is assumed to be greater than the utility of £1000 loss. For convenience let us ascribe the number zero to the loss of £1000 and the number 50 to the gain of £5000. Now, since our decision maker has said that his certainty equivalent for the standard gamble is £400 we can calculate his relative utility of a £400 gain as the average of 0 and 50 utils, i.e.

$$(\tfrac{1}{2} \times 0) + (\tfrac{1}{2} \times 50) = 25 \text{ utils.}$$

We now have relative utility numbers for three different sums of money, and these may be plotted on a graph such as Fig. 4.10. However, in order to be reasonably confident about the shape of the curve it is desirable to have more than three points. This may be achieved by asking the decision maker to state his certainty equivalents for other gambles, which may now include a gain of £400 as one of the outcomes. For instance one gamble offered might be a 50:50 chance of a gain of £5000 or a gain of £400. Suppose that the decision maker's certainty equivalent is £2000. We then estimate his relative utility of £2000 as

$$(\tfrac{1}{2} \times 25) + (\tfrac{1}{2} \times 50) = 37\tfrac{1}{2}.$$

[8] Alternatively one might ask him: 'given the choice of option (a) the certainty of receiving £2000 and option (b) a gamble in which the two possible outcomes are a gain of £5000 or a loss of £1000, what is the probability of gaining the £5000 at which you would be indifferent between the two options?' His answer could then be used in a similar way to that discussed above to arrive at the relative utility of £2000.

Now, £2000 may be offered as one of the prizes in a standard gamble, and so on.

The process of estimating additional points may be continued either until the curve can be drawn with confidence or until the decision

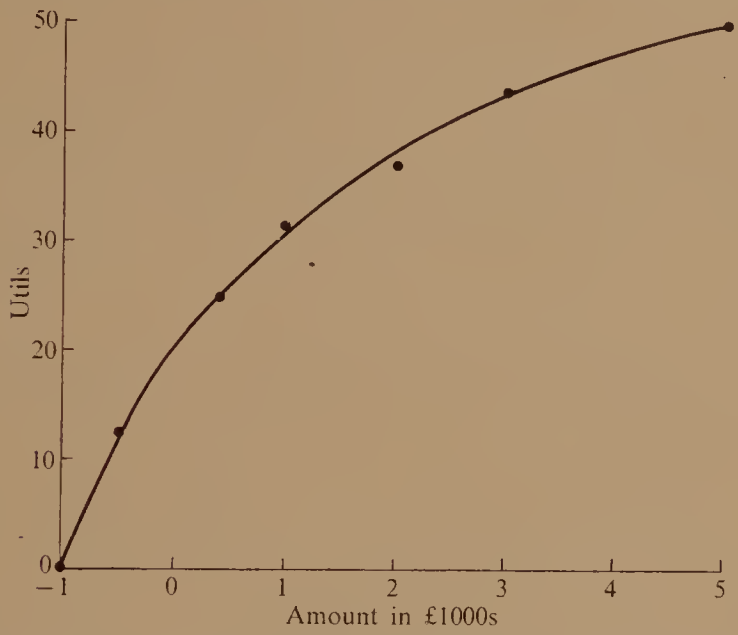


FIG. 4.10. Utility curve for monetary gains and losses. The curve is particular to an individual decision maker at a given point in time. The rate of change of slope of the curve indicates his degree of risk aversion.

TABLE 4.10

Relative utilities of different sums of money

Amount (£)	Relative utility (utils)
−£1000	0
−£ 500	$12\frac{1}{2}$
+£ 400	25
+£1000	$31\frac{1}{4}$
+£2000	$37\frac{1}{2}$
+£3000	$43\frac{3}{4}$
+£5000	50

maker finds the questioning too tedious to continue. We assume that five intermediate points have been evaluated as shown in Table 4.10. In practice few farmers or other individuals will make such consistent choices as our hypothetical decision maker, so that the points will not

usually lie as close to the utility curve. Indeed, some people find it difficult to treat such imaginary gambles seriously at all. However many successful attempts to estimate individual utility curves have been made.[9]

If enough points can be estimated and the decision maker is sufficiently consistent in his choices for a curve to be drawn, this curve may be used to evaluate *any* sum of money within the interval covered; in this example between −£1000 and + £5000. Having constructed the curve there is no need to ask the decision maker any more questions; all financial outcomes may be evaluated in utils by using Fig. 4.10. Choices between alternative plans or policies are then made so as to maximize the expected utility for the decision maker. For illustration, four alternative plans are compared in Table 4.11. Plan A has the highest EMV of the four but

TABLE 4.11

Alternative plans evaluated in terms of expected utility

	Good years revenue (£)	Poor years revenue (£)	EMV (£)	Variance	Good years utility (utils)	Poor years utility (utils)	Expected utility
Plan A	5000	1500	3250	3 062 500	50	35	42·5
Plan B	3250	2750	3000	62 500	45	42·5	43·7
Plan C	4500	1000	2750	3 062 500	48	31	39·5
Plan D	2750	2750	2750	0	42·5	42·5	42·5

also has the highest variance. Plan D has the minimum variance but also the joint-lowest EMV. When the alternative outcomes are evaluated in utils using Fig. 4.10 it turns out that plan B has the highest utility for our decision maker and is therefore his best plan. Note also that although plans C and D have the same EMV of £2750, plan D with the certain outcome has the higher expected utility.

Similarly the utility curve may be used to evaluate any set of risky outcomes, for instance those of the three plans in Table 4.3 or those appearing on the right-hand side of a decision tree. However, caution must be exercised in using this approach to choosing between risky alternatives for several reasons.

First, it should be noted that only total net income or total profit can be evaluated in utils. The basic assumption of diminishing marginal utility of money implies that the utility of 100 tonnes of grain sold at £20

[9] See for instance, Officer, R. R. and Halter, A. N. (1968). Utility analysis in a practical setting. *Am. J. agric. Econ.* **50**, 257.

per tonne is less than 100 times the utility of £20. Thus we cannot multiply quantity by value per unit in utils to arrive at total utility. Similarly the utility of £1000 profit is not the same as the utility of £3000 revenue minus the utility of £2000 costs, because the marginal utility of money diminishes.

Secondly, the utility numbers derived from the curve represent the relative utility of money for a particular decision maker at a particular point in time. Since the utility curve is personal and subjective, it is not possible to compare the utility of a given sum of money to different individuals. It may be convenient for us to ascribe the number 50 utils to a gain of £5000 for several different individuals but this decision is quite arbitrary. It must not be assumed that all these individuals will derive the same utility or satisfaction from £5000. Thus a separate curve must be constructed for each farmer. Indeed, in so far as a single individual's attitudes to additional income and risk taking vary over time, it may be necessary to construct a new utility curve for him from time to time.

Thirdly, the assumptions on which this approach is based may not be valid. For instance it is assumed that the relative utility of a given sum of money is independent of the probability of receiving it. Some psychologists question this assumption, pointing out that outcomes with very low probabilities of occurrence may be ignored altogether, meaning they are ascribed zero utility. Then again it is necessary to assume that the utility curve is continuous, otherwise the estimation procedure breaks down. Thus it cannot be used where the farmer has a hierarchy of objectives, which implies a discontinuous ranking of alternative plans. Whether the expected utility approach or the multiple objective approach is the more realistic representation of the way decisions are made remains an open question.[10]

Uncertainty and subjective probability

In reality most decisions are made under conditions of uncertainty; i.e. without knowledge of the relative frequency distributions of prospective outcomes. Indeed, some would argue that *all* decisions are made under uncertainty since even when relevant frequency distributions have been observed in the past it is not safe to assume that conditions will

[10] One study carried out in America suggests that the expected utility maximization hypothesis gives the best predictions; see Lin, W., Dean, G. W., and Moore, C. V. (1974). An empirical test of utility versus profit maximization in agricultural production. *Am. J. agric. Econ.* **56**, 497–508.

remain the same in the future if the proposed plan is put into operation. For instance, let us put ourselves in the position of a farmer who is considering the purchase of a flock of breeding ewes. Now suppose we are fortunate enough to have precise data, such as those presented in Tables 4.1 and 4.2, recorded recently on a similar farm for the same breed of sheep and same system of management that we propose. Is the expected revenue per ewe of £13.82, based on these figures, the best estimate to use in our own calculations? Not necessarily, since the age and genetic make-up of our ewes may differ, the stockmanship of our shepherd may differ, and prices may well have changed since the data were recorded. This does not mean that the available information is useless but it does mean that it should be used with caution and possibly modified in the light of our personal judgement and experience.

Despite the lack of information, decisions *are* made under uncertainty. Now it may be that decision makers, either consciously or subconsciously, use their judgement to assess the likelihood of different events occurring. From this point of view subjective probability (also known as personal or judgemental probability) may be interpreted as a state of mind: the relative strength of a person's belief that a given event will occur. Naturally, in establishing these subjective probabilities, the person will take account of any relevant data which are available from similar situations but he may also rely on 'educated' guesses or intuition.

Like relative utility, subjective probability may be evaluated in terms of gambling odds. For instance if our prospective sheep farmer is willing to bet 3 to 1 that he will earn a revenue of at least £14 per ewe, he is assigning a probability of 3 out of 4, or 0·75, to this event. Care must be taken in using this approach, however, if the marginal utility of money is not constant for the farmer. Thus suppose the farmer says he is willing to bet £300 against £100 that he can earn a return of at least £14 per ewe. This means he is willing to pay out £300 if he is wrong in exchange for a gain of £100 if he is right. If his utility curve for money is identical with that given in Fig. 4.10 we may read off the utility of a payout of £300 as the difference between the utilities of zero and − £300 which is $20 - 16 = 4$ utils. Similarly the utility of a gain of £100 is $21\frac{1}{2} - 20 = 1\frac{1}{2}$ utils. Hence his true subjective probability is

$$\frac{4}{4 + 1\frac{1}{2}} = \frac{8}{11} = 0{\cdot}73,$$

not

$$\frac{300}{300 + 100} = 0{\cdot}75.$$

Once probabilities are assigned to all possible outcomes of the alternative plans or actions, then the decision procedure is the same, regardless of whether the probabilities are objective or subjective. Thus subjective probabilities may be used to estimate EMVs and variances, or to determine expected utilities for the purpose of choosing between alternatives. Consequently, for practical purposes, there is no fundamental difference between risk and uncertainty. Indeed some writers refer to uncertainty as 'subjective risk'.

In a situation of subjective risk it is quite likely that different individuals faced with the same evidence will assign different probabilities to the same event and there is no absolute, objective way of judging who is right and who is wrong. However, an individual may make incoherent probability assignments, which do not conform with the basic rules of probability. For instance he may assess the probability of earning £14 or more per ewe at 0·73, and the probability of earning less than £14 per ewe at, say, 0·5. These assignments are incoherent since they add up to more than 1. This would of course raise serious problems in the calculations of expectations and variances but the individual may revise his estimates if the incoherence is pointed out to him.

Subjective probability estimates can be applied to any situation, whether or not any objective measurements are available. When there is no experience whatsoever on which to base estimates, equal probabilities may be assigned to all possible outcomes. Indeed the so-called principle of insufficient reason implies that if there is no reason to suppose that one outcome is more likely than another it is logical to treat them all as equally likely. Naturally these estimated probabilities may be wrong for instance if one assumed that single, twin, and triplet lamb births were equally likely, but this is only to be expected given the conditions of near total ignorance. This principle suffers from the disadvantage that it is not clear in advance what unknown possibilities are to be considered equally probable. For instance, in the complete absence of meteorological data we might assume wet and dry years to be equally probable and assign them each the probability of $\frac{1}{2}$, as in Tables 4.4 and 4.5. However, if we now admit the possibility of intermediate years which are neither very wet nor very dry, we now have 3 possible outcomes and the probability of each is reduced to $\frac{1}{3}$ as in Tables 4.6, 4.7, and 4.8. We see, then, that by a simple act of reclassification, the probabilities of occurrence of wet and dry years are changed. Nevertheless, the principle may be useful in situations where there is no information available.

In some cases a decision maker may decide to collect information rather than embark on a productive activity immediately. Alternatively

he may decide to go ahead with a particular productive activity on a trial basis, again with a view to collecting more information; i.e. he is learning. Naturally the decision maker will revise his subjective probability estimates in the light of this new information. Thus if our sheep farmer finds that he earns a total revenue per ewe of over £14 in, say, the first three years of production he will undoubtedly raise his subjective probability estimate for this event.[11] As a result, while different persons may initially assign different probabilities to the same event, these personal probabilities will tend to converge on the objective probability as the number of observations increases.

The reader may well be wondering by this stage whether any farmer ever consciously goes through all the procedures discussed above; i.e. of estimating subjective probabilities, then using these to calculate either EMVs and variances or expected utility indices for each alternative plan or policy, before reaching a decision. No one would claim that this is an accurate, or even an approximate, description of how most decisions really are made. Even if it were, we could not make accurate predictions of farmers' decisions in general because subjective probability estimates (beliefs) and attitudes to risks (objectives) vary from one farmer to another. However, it is a logical method for analysing large and complex problems, which may be used by individual farmers and their advisors in dealing with planning problems on the individual farm.

Games against nature

Game theory originated in the logical analysis of competitive parlour games between two or more players. In such games uncertainty arises because neither player knows what decisions his opponent will make, although the theory may enable one to predict what decisions he is likely to make. However, the problem of individual decision making under complete uncertainty may be considered as a one-person game against nature. Nature is assumed to behave in a totally unpredictable manner.

Much of the terminology is carried over from pure game theory. Thus the decision maker is often referred to as the 'player', each alternative plan or activity open to him is known as a 'strategy' and each possible

[11] So-called Bayesian statistical methods, named after the Reverend Thomas Bayes, whose original paper on this subject was published in 1763, may be used to calculate revised (posterior) probabilities from given initial (prior) probabilities in the light of new statistical data. These prior probabilities may be subjective. See, for instance, Eidman, V. R., Dean, G. W., and Carter, H. O. (1967). An application of statistical decision theory to commercial turkey production. *J. Fm Econ.* **49**, 852.

final outcome is known as a 'pay-off'. When applied to farming decisions 'nature' takes the place of the second player and each of his strategies becomes a 'state of nature'. The pay-offs for all states of nature and all strategies are set out in a 'pay-off matrix', as in Table 4.12. To allow for possible variation in the marginal utility of money all pay-offs are expressed in utils. The matrix then applies only to a particular individual decision maker.

It may be noted that Tables 4.4–4.8 may also be viewed as pay-off matrices with pay-offs expressed in money terms. Incidentally Tables

TABLE 4.12

Pay-off matrix for an individual farmer (amounts in utils)

		States of nature			
		Very wet	Wet	Dry	Very dry
Farmer's strategies	1. Dairying	20	22	15	5
	2. Arable	15	15	17	10
	3. Market gardening	5	40	5	0
	4. Small livestock	10	30	6	5

4.5 and 4.7 illustrate a situation where one strategy is dominated by another, which means that the other strategy will be preferred under all circumstances and for all reasonable criteria. Thus in Table 4.5 we see that the pay-off from grain production (strategy 1) is always equal to or greater than the pay-off from root production (strategy 2) whatever the state of nature. Similarly in Table 4.7 root production is seen to dominate grain production. It is *never* to a player's advantage to use a dominated strategy.

Where no strategies are dominated, as in Table 4.12, various alternative decision rules have been suggested for choosing between them. None of these decision rules is entirely satisfactory, in part because they do not take all possible states of nature into account. It is possible to find cases where they give solutions which are obviously unacceptable. Nevertheless some of these decision rules have been applied in farm planning so they will be discussed briefly below. Note first that if we applied the principle of insufficient reason to the data in Table 4.12 we would select strategy 1 (dairying) since this would yield the highest

expected utility of 15·5 utils as compared with 14·25, 12·5, and 12·75 utils for strategies 2, 3, and 4 respectively.

The maximin criterion

We have already mentioned maximin policies in discussing choices under risk. However, the worst possible outcome for any plan or strategy can be defined without any prior knowledge of the probabilities of different outcomes occurring. Hence the maximin criterion may be used to choose between alternative strategies under uncertainty. Referring to Table 4.12 we see that the worst possible pay-off for dairying (strategy 1) is 5 utils, obtained under very dry conditions. Indeed in this particular matrix the minimum pay-off for every strategy is found in the last column, representing very dry conditions. The best of these worst possible pay-offs (the maximin) occurs in the strategy 2 row, so that, guided by the maximin criterion, the farmer would choose cereal production. By making this choice he can be sure of receiving a pay-off of at least 10 utils every year.

The maximin strategy is the obvious one to choose in a game against an opponent who is out to win and therefore to keep your gains as small as possible. But in a game against nature you would *not* normally expect nature always to do its worst. Thus in decision making the maximin criterion leads to over cautious behaviour. It implies either extreme pessimism, i.e. a subjective probability of 1 for the worst possible outcome, or such extreme risk aversion that all but the worst outcomes are ignored. This may, of course, be appropriate in harsh conditions where survival is precarious.

Maximin mixed strategy

In some cases, though not with the pay-offs in Table 4.12, a 'mixed strategy' which is a combination of pure strategies may give a higher minimum pay-off than any pure strategy alone. It is not always possible to combine pure strategies in any one production period, but when this can be done, the decision maker is able to *ensure* that his annual income does not fall below this maximin level.[12] For example if Table 4.4 is viewed as a pay-off matrix we see that the mixed strategy of $\frac{2}{3}$ grain and $\frac{1}{3}$ roots ensures a minimum pay-off of £23.33. This may be compared

[12] Many business decisions differ from games in this respect, since in a game it is not possible to play more than one pure strategy at a time. Strategies can then be mixed only by switching from one to another for successive plays of the game. This must be done at random to confuse the other player. The maximin mixed-strategy pay-off is then an expected value; the player cannot be sure of getting the same amount in every play of the game.

with the minimum pay-off of £20 for the pure strategy of grain production alone or of £10 for the pure strategy of root production alone.

The determination of the optimal combination of grain and roots for this maximin mixed strategy is illustrated in Fig. 4.11. The line labelled 'wet years' shows how the actual gross margin per hectare obtained in wet years varies according to the proportions in which grain and roots are combined. Thus with *no* roots the gross margin in wet years from

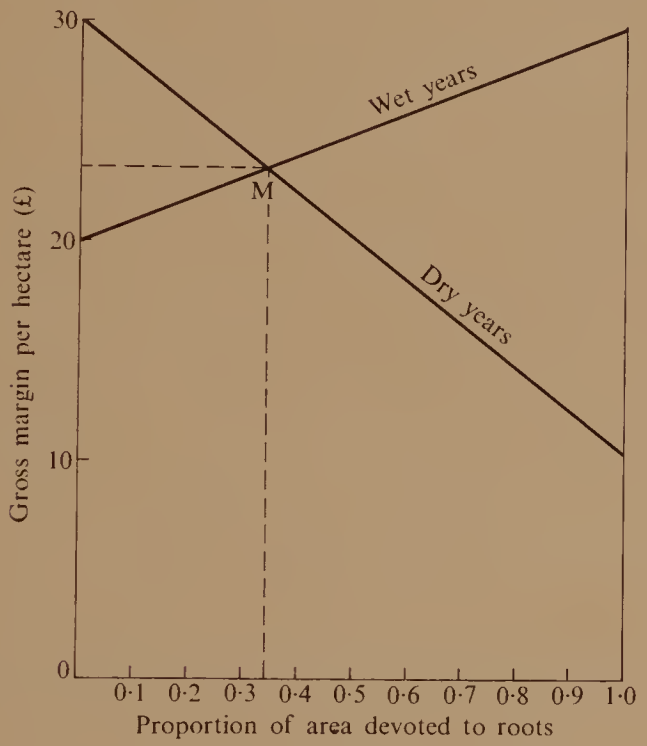


FIG. 4.11. The optimum maximin, mixed strategy. Since the return (pay-off) from grain is high in dry years whereas the return (pay-off) from roots is high in wet years, a combination of the two activities (a mixed strategy) will yield a higher minimum return in all years than will either activity alone (pure strategies). In this case the maximin mixed strategy consists of a combination of $\frac{1}{3}$ roots and $\frac{2}{3}$ grain yielding a gross margin of £23.33 in all years (point M).

grain alone is £20, but if roots are introduced the *average* gross margin per hectare in wet years *rises*. When all the area is under roots the gross margin per hectare in wet years is £30. Similarly, the line labelled 'dry years' shows how the *average* gross margin per hectare *falls* as the proportion of roots is increased. The largest of the worst pay-offs, the maximin combination, is found where the two lines interesect, that is when 0·33 or $\frac{1}{3}$ of the area is devoted to roots. With less than $\frac{1}{3}$ of the

area devoted to roots the minimum pay-off occurs in wet years and increases as the proportion of roots is increased. Beyond this point, however, the minimum pay-off occurs in dry years and falls as the proportion of roots rises. By growing $\frac{1}{3}$ roots and $\frac{2}{3}$ grain the farmer can be sure of obtaining a gross margin of £23·33 per hectare every year as shown in Table 4.4. In this very simple case the maximin solution is identical with the minimum variance combination of grain and roots but this is not necessarily so when more then two states of nature are involved.

This problem could be solved by linear programming using the matrix of Table 4.9, but omitting the constraint levels of £1100 for survival needs and the expected values. The new objective is to maximize the minimum return, which is also treated as a constraint level. Production under each state of nature must be equal to or greater than this maximin return.[13] In fact it may be noted that point B in Fig. 4.8, which represents a 3:1 combination of grain and roots is a maximin solution for the present example. Much larger problems with many alternative activities, many states of nature and many resource and other constraints may be dealt with in this way. Indeed a minimum acceptable level of EMV may be introduced as a constraint, thereby providing an alternative method of arriving at the EMV–lower-limit relationship depicted in Fig. 4.9.

We can tell from a consideration of the pay-off matrix whether a mixed strategy will yield a higher maximin value than a pure strategy. If the pure-strategy maximin value, which must be the smallest of its row, is also the largest of its column, it is called a saddle point. It is impossible to improve upon the solution by using a mixed strategy. Thus in Table 4.12, the pay-off of 10 utils from arable farming under very dry conditions is both the row minimum and the column maximum. It is therefore a saddle point and cannot be improved upon. However, in Table 4.4. the maximin pure strategy of grain production yields the worst pay-off of £20 in wet years. This pay-off is not the largest in its column because roots yield more in wet years. Hence a mixed strategy will increase the minimum pay-off.

The partial optimist criterion

A complete pessimist always expects the worst possible outcome and bases his decisions on the maximin criterion. At the other extreme, a complete optimist considers only the best possible pay-off for each

[13] For proposed methods of incorporating the maximin and other game-theory criteria into the linear programming framework, see Hazel, P. B. R. (1970). Game theory—an extension of its application to farm planning under uncertainty. *J. agric. Econ.* **21**, 239–52.

strategy and chooses the highest of these. It has been suggested that the typical decision maker is a *partial* optimist who therefore bases *his* decisions on the weighted average of the best and worst pay-offs for each strategy. Thus for each person there is supposed to be a weighting factor (α) between 0 and 1 which is a measure of his relative pessimism. Let us assume the weighting factor (α) is 0·6, which implies the decision maker is 0·6 pessimist and $1 - 0{\cdot}6 = 0{\cdot}4$ optimist. Each of the strategies in Table 4.12 are evaluated as follows:

1. Dairying $0{\cdot}6\ (5) + 0{\cdot}4\ (22) = 11{\cdot}8$ utils
2. Arable $0{\cdot}6\ (10) + 0{\cdot}4\ (17) = 12{\cdot}8$ utils
3. Market gardening $0{\cdot}6\ (0) + 0{\cdot}4\ (40) = 16$ utils
4. Small livestock $0{\cdot}6\ (5) + 0{\cdot}4\ (30) = 15$ utils

Thus if his α value is 0·6 our decision maker will choose strategy 3, market gardening on the basis of this criterion.

The problem of estimating a man's α value is clearly akin to the problem of estimating subjective probabilities. However, the use of the partial-optimist criterion is less satisfactory since the probabilities of outcomes other than the best and worst are ignored. Mixed strategies may be selected with the partial-optimist criterion but not so readily as with the maximin criterion. While mixing strategies may increase the minimum pay-off, it tends to reduce the maximum pay-off so the weighted average of these may not rise when the minimum rises.

The minimax regret

Instead of choosing on the basis of actual pay-offs the decision maker may be more concerned with the opportunity cost of making the wrong (sub-optimal) decision. For instance, referring to Table 4.12 again, let us assume that our farmer selects strategy 3, market gardening. Then clearly if it turns out to be very wet he will have made a sub-optimal decision since his pay-off will only be 5 utils where he could have obtained 20 utils from dairying. Thus we can express his opportunity cost or 'regret' in these circumstances as $20 - 5 = 15$ utils. If on the other hand it turns out to be wet, his regret will be zero since market gardening yields the largest pay-off for this state of nature. His regrets under dry and very dry conditions would be $17 - 5 = 12$ utils and $10 - 0 = 10$ utils respectively.

Now if the farmer knows his pay-off matrix in advance then he can calculate what his regrets would be for every strategy under every state of nature. These calculations yield a regret matrix as shown in

Table 4.13. To protect himself against excessive opportunity cost the farmer may now choose the strategy which will minimize the maximum regret. Thus for dairying we see that the largest regret is 18 utils; for arable it is 25 utils, for market gardening it is 15 utils and for small livestock 11 utils. Hence the least of these largest regrets is ensured by choosing small livestock, strategy 4.

This criterion only takes account of the largest regret figure in any row; all other outcomes are ignored. Once again mixed strategies may be selected on the basis of the minimax regret criterion.

Note that Table 4.12 was so constructed that different strategies are

TABLE 4.13

Regret matrix for the same farmer (amounts in utils)

		States of nature			
		Very wet	Wet	Dry	Very dry
Farmer's strategies	1. Dairying	0	18	2	5
	2. Arable	5	25	0	0
	3. Market gardening	15	0	12	10
	4. Small livestock	10	10	11	5

selected on the basis of different criteria. Strategy 1, dairying, gives the highest expected utility for equal probabilities; strategy 2, arable, gives the maximin pay-off; strategy 3, market gardening, gives the highest value for a pessimism–optimism weighting of 0·6:0·4; strategy 4, small livestock, gives the minimax regret. The choice of criterion and hence the choice of strategy must be somewhat arbitrary. However, there is fairly wide-spread agreement that if subjective probabilities can be estimated for all prospective outcomes then this is a more rational way of dealing with uncertainty.

Rules of thumb

All the theories we have discussed are somewhat unrealistic; very few decisions are actually made in the ways described. However, it is argued that the purpose of these theories is not descriptive but normative; they prescribe what farmers ought to do in order to achieve certain objectives. Even this argument may be questioned. On the one hand decision theories simplify reality in assuming relatively few objections

relatively few alternative policies, and relatively few alternative outcomes. The real world is far more complex than this: indeed, so much so that even with the aid of powerful computers, most real-life choices still lie beyond the reach of optimizing techniques, unless the situations are simplified by drastic approximations. On the other hand the suggested approaches to decision making incur considerable costs in terms of time and effort if not money. Time and effort are involved in understanding these approaches, collecting information, listing alternatives, predicting outcomes, estimating probabilities, and evaluating alternatives. Thus for any decision there must come a point where the extra cost of greater refinement in the decision making process must exceed the extra benefit. The economic optimum approach to decision making might well be a very unsophisticated, rough-and-ready one.

In practice farmers and other businessmen may not be concerned with finding the optimum or best possible choice of action. Instead they have certain limited objectives or aspiration levels, and any plan which satisfies these objectives is likely to be accepted. Such behaviour is described as 'satisficing' in contrast with the 'optimizing' or 'maximizing' behaviour we have assumed so far. As long as existing policies continue to satisfy the farmer's objectives he will be unwilling to change. Many of his productive activities are dictated by habit or custom, though if it proves easy to achieve existing levels of aspiration these levels may gradually be revised upwards. When performance falls short of the level of aspiration, however, then the farmer will be induced to search for new alternative production policies. However the search will continue only until a satisfactory solution is found, not until an optimum is found. At the same time, the level of aspiration may be adjusted downward until goals reach levels that are more readily attainable.

In his search for a satisfactory plan of action the farmer may make use of simple rules of thumb, known as 'heuristics', rather than any formal planning procedure. Several such rules of thumb are applied in farming as precautions against risk and uncertainty. These include:

(1) choice of reliable enterprises;
(2) diversification;
(3) discounting for risk;
(4) maintaining flexibility;
(5) maintaining reserves;
(6) insurance;
(7) contracting prices in advance.

All of these are likely to incur a cost so some expected profit is forgone, but this must be set against the benefit of greater security.

1. *Choice of reliable enterprises*

Farmers, even though they may not know the whole range of possible outcomes and probabilities, are aware that some enterprises are more reliable than others. The relative yield variation of intensive livestock, such as pigs and poultry, is generally thought to be less than that of sheep and beef cattle for instance. Cereal yields generally vary less in relation to their average than do those of root crops. Similarly there is widespread recognition that the prices of some products are more stable than those of others. The risk avoider may choose the more reliable enterprise, even when he expects that activities with variable outcomes would give him a higher long-run average income.

Risk avoiders will be cautious in adopting innovations. Even when trials have shown that the innovation can yield high and stable returns, a farmer cannot be sure that it will be successful under the conditions in his locality. He may prefer enterprises which he and his neighbours have found to be reliable rather than experiment with innovations.

2. *Diversification*

This rule is widely recommended as a means of reducing income variability, but as we have seen it may increase risk. Thus it is only to be recommended as a risk precaution when it is known that returns from the enterprises involved are *not* positively related.

3. *Discounting for risk*

This implies producing less than the economic optimum level of output on average in order to reduce losses in unfavourable seasons. We may illustrate this with the hypothetical example from Table 3.1 (p. 76). However, we now assume the total, average, and marginal products per unit of labour are expected values, and that the results can vary by 50 per cent either side of the average, as shown in Table 4.14. This means that in poor or unfavourable years the farmer makes losses but these are balanced by larger-than-average profits in good years. Over a long period expected profit is still maximized by using about 4 units of labour, as shown in Chapter 3. However, in poor years, the use of this amount of labour would result in a loss of £11.50. The loss in poor years is minimized at £9.50 by using only 3 units of labour. On the basis of the maximin criterion, this strategy of applying only 3 units of labour would be chosen, although it would mean forgoing

TABLE 4.14

Risk and the economic optimum

Units of labour input	Total cost (£)	Poor years			Good years			Expected value		
		Value of total product (£)	Profit (£)	Value of marginal product (£)	Value of total product (£)	Profit (£)	Value of marginal product (£)	Value of total product (£)	Profit (£)	Value of marginal product (£)
0	10	0	−10·0		0	−10·0		0	−10	
				3·5			10·5			7
1	15	3·5	−11·5		10·5	−4·5		7	−8	
				6·5			19·5			13
2	20	10·0	−10·0		30·0	10·0		20	0	
				5·5			16·5			11
3	25	15·5	−9·5		46·5	21·5		31	6	
				3·0			9·0			6
4	30	18·5	−11·5		55·5	25·5		37	7	
				1·5			4·5			3
5	35	20·0	−15·0		60·0	25·0		40	5	

Expected values are taken from Table 3.1. Total product is assumed to be 50 per cent lower than average in poor years and 50 per cent higher in good years. Cost per unit of input is assumed to remain constant at £5. Note that a negative profit is a loss, so profit is maximized or loss minimized at the economic optimum, where the value of the marginal product is approximately equal to the unit factor cost.

some profit (in this case £7 − £6 = £1) on average. The majority of farmers who are neither maximiners nor EMV maximizers would probably choose a level of labour input between 3 and 4 units. In other words they would use somewhat less input and produce somewhat less output than the economic optimum levels.

4. *Maintaining flexibility*

This implies being prepared to make new decisions from day to day in the light of changing conditions, rather than preparing a comprehensive plan in advance and sticking to it rigidly. Generally, a flexible farm system is one which can be changed more quickly and at lower cost than an inflexible system. Flexibility may be maintained by choice of enterprises with a quick turnover, by using general-purpose buildings and machines rather than specialized ones, and by avoiding fixed costs. It is easier and cheaper to expand or contract an enterprise such as pig or poultry keeping with a relatively short production period than it is to vary the scale of a pedigree-cattle breeding enterprise, provided that specialized buildings and equipment are avoided. Likewise annual crops are likely to provide greater flexibility than perennial tree crops. Also, a farm system is kept flexible by using temporary hired labour rather than regular workers, by using labour-intensive methods rather than specialized machinery, and by hiring on contract rather than owning machines. However, this flexibility is likely to have a cost in terms of a certain amount of profit forgone on average.

5. *Maintaining reserves*

This may be viewed as another way of maintaining flexibility. Thus many farmers maintain a larger complement of machinery and equipment and sometimes a larger labour force than is strictly necessary in an average year, in order to be prepared for adverse conditions. At the same time reserves of food or cash may be kept to tide the farm family over periods of adverse yields or prices.

6. *Insurance*

Insurance may cover major risks such as the death of the farmer or some member of his family; it may be used for sickness and accidents which disable the farmer or his workers and for fires or other hazards which can destroy capital items such as buildings, breeding and fattening stock, cars, lorries, and other machines. In some places it is possible to insure against the loss of crops by hail or hurricanes. Clearly farmers,

and others who buy insurance, forgo a certain amount of income each year (the annual premium) in return for the security offered.

7. *Contracting prices in advance*

In some cases price uncertainty may be reduced by making advance contracts for the sale of produce and purchase of inputs. However, where contracts are made for more than one year in advance the parties involved generally reserve the right to re-negotiate the terms from time to time, so variation and uncertainty cannot be eliminated entirely. For instance, rents and wages are generally fixed by contract but landlords and workers may ask for increases in their contractual rents and wages periodically. Some production takes place almost entirely under contractual agreement, particularly in the poultry industry. Farmers provide housing, labour, and some management while the outside party, usually a livestock feed company, supplies chicks and feed and pays the farmer an agreed price for the product. For obvious reasons the development of such a system is referred to as vertical integration. Again, it is possible that the farmer could make more profit on average over a period of years if he operated on his own but through the contractual agreement he obtains greater security and has fewer worries.

Summary

1. The outcomes of real-world decisions cannot be predicted with absolute certainty. It is therefore impossible to select, in advance, the policy which will maximize the *actual profit* which will be obtained by the end of a given production period.

2. In a situation of risk, in which probabilities and therefore expectations can be estimated, decisions may be made so as to maximize *expected profit* or expected monetary value (EMV). The optimizing theory of earlier chapters may be viewed in this light. However, some examples have demonstrated that one can be seriously misled by using average values in place of the previously assumed certain values in this optimizing theory (fallacy of averages).

3. Most people are risk averse, which means that they are willing to forgo some expected profit in exchange for greater security. Risk aversion may be taken into account:

(a) by evaluating alternative policies in terms of EMV *and variance*,
(b) by evaluating alternative policies in terms of EMV and *minimum* monetary value;
(c) by evaluating alternative policies in terms of the individual's *expected utility*,

Alternatives (a) and (b) involve two choice criteria which must be reconciled. It may be possible to rank the objectives in order of priority (lexicographic ordering) or replace the two measures by a single certainty equivalent to be maximized. Alternative (c) is a special case of a certainty equivalent. The decision criteria are reduced to a single objective of maximizing expected utility.

4. Under uncertainty, probabilities are unknown. A rational approach is to assign subjective (personal) probabilities and then to apply the method for decision making under risk. Alternatively a simple criterion such as the maximin, the partial-optimist or the minimax regret criterion may be used, though there is no clear logical justification for doing so in making business decisions.

5. In practice decision makers may prefer to rely on some basic rules of thumb (heuristics).

Mathematical appendix

Probability

1. *Definitions.*

Trial: any well-defined action upon which an observation can be made or a measurement taken.

Outcome: Each possible, distinct result of a trial; thus we define outcomes as being mutually exclusive (not more than one of them can occur in a single trial) and mutually exhaustive (together they account for all possible results).

Event: any well-defined set of outcomes.

Probability: a measure of the likelihood of occurrence of a particular outcome or event.

Objective (or relative frequency or statistical) probability:

$$P(\mathrm{A}) = \frac{n}{N},$$

where $P(\mathrm{A})$ = probability of event A,

n = number of times event A occurs (successes),

N = total number of trials, which must be infinitely large for the measure to be exact. Thus in practice all such measures are only estimates.

2. *Basic assumptions.*

(a) The probability of every conceivable outcome must be greater than or equal to zero.

(b) The sum of the probabilities of all conceivable outcomes is 1.
(c) The probability of event A is the sum of the probabilities of all outcomes making up event A.

3. It follows from these assumptions that:
(a) $0 \leqq P(A) \leqq 1$,
For example, the probability of ewe bearing twins = 0·4 (see Table 4.1).
(b) $P(A') = 1 - P(A)$ where $P(A')$ is the probability that A does not occur. For example, the probability of a ewe not bearing 1 lamb $= 1 - 0{\cdot}4 = 0{\cdot}6$.
(c) For two mutually exclusive events A and B,
$P(A + B) = P(A) + P(B)$,
where $P(A + B)$ = probability of either A or B.
For example, the probability of a ewe bearing either 1 or 2 lambs

$$= 0{\cdot}5 + 0{\cdot}4 = 0{\cdot}9.$$

(d) Generally, for any two events A and B,

$$P(A + B) = P(A) + P(B) - P(AB),$$

where $P(AB)$ = joint probability of both A and B occurring together and $P(A + B)$ = probability of either A or B *or both*. For example, the probability of a ewe bearing either 1 or 2 lambs = 0·9 (from above); the probability of a multiple birth (i.e. 2 or 3 lambs) = 0·49 (Table 4.1); the probability of both '1 or 2 lambs' and 'multiple birth' (i.e. probability of twin lambs) = 0·4 (Table 4.1). Hence the probability of a ewe bearing either 1 or 2 or 3 lambs

$$= 0{\cdot}9 + 0{\cdot}49 - 0{\cdot}4 = 0{\cdot}99.$$

4. The conditional probability that event B will occur given that event A has occurred is written $P(B/A)$. We then have

$$P(AB) = P(A) \,.\, P(B/A), \tag{4.1}$$

which is called the general multiplication rule.

For example, the probability of a ewe bearing a single lamb = 0·5; the conditional probability of selling a single lamb for £10 = 0·6. Hence the probability of £10 revenue per ewe (i.e. a single lamb selling for £10)

$$= 0{\cdot}5 \times 0{\cdot}6 = 0{\cdot}3.$$

Note that rearrangement of eqn (4.1) gives

$$P(B/A) = P(AB)/P(A), \tag{4.2}$$

which is a mathematical definition of conditional probability.

5. Events A and B are said to be *independent* if

$$P(\mathrm{B/A}) = P(\mathrm{B}).$$

Substitution in eqn (4.1) gives

$$P(\mathrm{AB}) = P(\mathrm{A}) \,.\, P(\mathrm{B}),$$

which is called the *special multiplication rule* (for independent events).

For example, the probability of a ewe bearing a single lamb = 0·5; the probability of a lamb selling for £10 = 0·4. Therefore the probability of £10 revenue per ewe

$$= 0{\cdot}5 \times 0{\cdot}4 = 0{\cdot}2.$$

Expectations

1. In a simple gamble the expected gain $E(G)$ is given by the value of the prize G multiplied by the probability of winning $P(G)$. Note that if the gambler does not win he gains nothing, thus

$$\begin{aligned} E(G) &= G \,.\, P(G) + 0(1 - P(G)) \\ &= G \,.\, P(G). \end{aligned}$$

2. More generally, for a discrete random variable a, the expected value $E(a)$ is given by;

$$E(a) = a_1 \,.\, P(a_1) + a_2 \,.\, P(a_2) + \ldots + a_n \,.\, P(a_n).$$

For a finite number of observations the expected value $E(a)$ is the same as the average (arithmetic mean) also written as $\bar{a}$.

For example, the expected revenue per ewe (from Fig. 4.3) is given by £(0 × 0·01) + (9·2 × 0·5) + (18 × 0·4) + (26·4 × 0·09) = £14.176.

3. If k is a constant then

$$E(a + k) = E(a) + k.$$

For example, let expected revenue per ewe $E(a)$ be defined as above = £14.176. Now let k = constant cost (including depreciation) per ewe = −£10. Then expected margin (revenue minus cost) per ewe is given by

$$\begin{aligned} E(a + k) = \pounds(-10 \times 0{\cdot}01) + (-0{\cdot}8 \times 0{\cdot}5) + (8 \times 0{\cdot}4) + \\ +(16{\cdot}4 \times 0{\cdot}09) = \pounds 4.176 = \pounds 14.176 - 10 = E(a) + k. \end{aligned}$$

4. If b is a second random variable, then

$$E(a + b) = E(a) + E(b).$$

For example, let expected revenue per ewe $E(a)$ be defined as above = £14.176.

Now let b_1 = cost per ewe with no lambs = −£5;

b_2 = cost per ewe with 1 lamb = −£8;

b_3 = cost per ewe with 2 lambs = −£12;

b_4 = cost per ewe with 3 lambs = −£14.

Then

$$E(b) = £(-5 \times 0{\cdot}01) + (-8 \times 0{\cdot}5) + (-12 \times 0{\cdot}4) + \\ + (-14 \times 0{\cdot}09) = -£10.11$$

and

$$E(a + b) = £(-5 \times 0{\cdot}01) + (1{\cdot}2 \times 0{\cdot}5) + (6 \times 0{\cdot}4) + \\ + (12{\cdot}4 \times 0{\cdot}09) = £4.066$$
$$= £(14.176 - 10.11) = E(a) + E(b).$$

5. Again let k be a constant, then

$$E(k \,.\, a) = k \,.\, E(a).$$

For example, let a = number of lambs per ewe as presented in Table 4.1:

$E(a) = (0 \times 0{\cdot}01) + (1 \times 0{\cdot}5) + (2 \times 0{\cdot}4) + (3 \times 0{\cdot}99) = 1{\cdot}57.$
Now let k = constant price per lamb of £8·8. Then

$$E(k \,.\, a) = £(0 \times 0{\cdot}01) + (8{\cdot}8 \times 0{\cdot}5) + (17{\cdot}6 \times 0{\cdot}4) + (26{\cdot}4 \times 0{\cdot}09) \\ = £13.816 = £8.8 \times 1{\cdot}57 = k \,.\, E(a).$$

6. If a and b are both random variables which are *independent*

$$E(a \,.\, b) = E(a) \,.\, E(b).$$

For example, let a = number of lambs per ewe as above. Then $E(a) = 1{\cdot}57$. Now let b = variable price per lamb such that

$$b_1 = £8 \quad ; P(b_1) = 0{\cdot}6$$
$$b_2 = £10 \; ; P(b_2) = 0{\cdot}4,$$

then

$$E(b) = £(8 \times 0{\cdot}6) = (10 \times 0{\cdot}4) = £8.8.$$

$E(a \,.\, b) = £13.816$ (see Table 4.2) $= £8.8 \times 1{\cdot}57 = E(a) \times E(b).$

The fallacy of averages

Note that all the above statements imply that expected values may be manipulated in the same way as certain outcomes. *But* if a and b are *not* independent, then

$$E(a \,.\, b) \neq E(a) \,.\, E(b).$$

For example, let a = lambs per ewe such that

a_1 = lambs per ewe in unfavourable seasons = 1

a_2 = lambs per ewe in favourable seasons = 2;

$P(a_1) = 0{\cdot}7$; $P(a_2) = 0{\cdot}3$.

Then

$$E(a) = (1 \times 0{\cdot}7) + (2 \times 0{\cdot}3) = 1{\cdot}3.$$

Let b = price per lamb; b_1 = price in unfavourable seasons = £10; and b_2 = price in favourable seasons = £8, where

$$P(b_1) = P(a_1) = 0{\cdot}7, P(b_2) = P(a_2) = 0{\cdot}3.$$

Then

$$E(b) = £(10 \times 0{\cdot}7) + (8 \times 0{\cdot}3) = £9.4.$$

Now, revenue per ewe is given by $a \,.\, b$. Thus

a_1b_1 = revenue per ewe in unfavourable seasons = £10;

a_2b_2 = revenue per ewe in favourable seasons = £16.

Hence expected revenue per ewe is given by

$$E(a \,.\, b) = £(10 \times 0{\cdot}7) + (16 \times 0{\cdot}3) = £11.8$$
$$\neq £9.4 \times 1{\cdot}3.$$

Thus in general it is not *safe to assume that*

$$E(f(a, b)) = f(E(a), E(b)).$$

This assumption may lead to the 'fallacy of averages'.

Variance and covariance

1. The variance of a random variable a is a measure of its variability or dispersion. It may be designated Var(a) and is defined by

$$\text{Var}(a) = (a_1 - \bar{a})^2 \,.\, P(a_1) + (a_2 - \bar{a})^2 \,.\, P(a_2) + \ldots$$
$$+ (a_n - \bar{a})^2 \,.\, P(a_n).$$

It can be shown that this is equivalent to

$$\text{Var}(a) = E(a^2) - [E(a)]^2$$

For example, referring to the example in the previous section *The fallacy of averages*, we have

$$\text{Var}(a) = (1 - 1{\cdot}3)^2 \times 0{\cdot}7 + (2 - 1{\cdot}3)^2 \times 0{\cdot}3 = 0{\cdot}21$$

or, using the second equation,

$$\text{Var}(a) = (1^2 \times 0{\cdot}7 + 2^2 \times 0{\cdot}3) - (1{\cdot}3)^2 = 0{\cdot}21.$$

2. The covariance of two random variables a and b is a measure of their joint variability. It may be designated Cov(ab) and is defined by

$$\text{Cov}(ab) = (a_1 - \bar{a})(b_1 - \bar{b}) \,.\, P(a_1, b_1) + (a_2 - \bar{a})(b_2 - \bar{b}) \,.\, P(a_2, b_2) + \ldots + (a_n - \bar{a})(b_n - \bar{b}) \,.\, P(a_n\, b_n),$$

where a_1 and b_1, a_2 and $b_2 \ldots a_n$ and b_n are paired observations. It can be shown that this is equivalent to

$$\text{Cov}(ab) = E(a \,.\, b) - E(a) \,.\, E(b).$$

For example, again referring to the example in the previous section, we have

$$\text{Cov}(ab) = (1 - 1{\cdot}3)(10 - 9{\cdot}4) \times 0{\cdot}7 + (2 - 1{\cdot}3)(8 - 9{\cdot}4) \times 0{\cdot}3 = -0{\cdot}42$$

or, using the second equation,

$$\text{Cov}(ab) = 11{\cdot}8 - (1{\cdot}3)(9{\cdot}4) = -0{\cdot}42.$$

3. Since

$$\text{Cov}(ab) = E(a \,.\, b) - E(a) \,.\, E(b),$$

by rearrangement we obtain

$$E(a \,.\, b) = E(a) \,.\, E(b) + \text{Cov}(ab).$$

Hence *except* when Cov(ab) = 0 (as would be the case, for instance, where a and b are independent).

$$E(a.b) \neq E(a) \,.\, E(b),$$

as stated above.

4. The correlation coefficient, designated r, is a scale-free (pure number) measure of the strength of linear association between variables. For two variables it is defined by

$$r = \frac{\text{Cov}(ab)}{\sqrt{\{\text{Var}(a) \,.\, \text{Var}(b)\}}}$$

For example, from points 1 and 2 above,

$$\text{Var}(b) = (10^2 \times 0{\cdot}7 + 8^2 \times 0{\cdot}3) - (9{\cdot}4)^2 = 0{\cdot}84.$$

Therefore

$$r = \frac{-0{\cdot}42}{\sqrt{(0{\cdot}21 \times 0{\cdot}84)}} = -1.$$

In fact with only 2 observations as in this example r must equal $+1$ or -1, but where there are more than 2 observations r usually lies somewhere between these limits.

5. If the same constant is added to all values of a random variable the variance is unaltered:

$$\text{Var}(a + k) = \text{Var}(a).$$

For example, referring to the example under the section *Expectations*, point 3,

a = revenue per ewe and k = constant cost per ewe.

$$\text{Var}(a) = (0^2 \times 0{\cdot}01) + (9{\cdot}2^2 \times 0{\cdot}5) + (18^2 \times 0{\cdot}4) + \\ + (26{\cdot}4^2 \times 0{\cdot}09) - (14{\cdot}176)^2 = 33{\cdot}69$$

$$\text{Var}(a + k) = (-10^2 \times 0{\cdot}01) + (-0{\cdot}8^2 \times 0{\cdot}5) + (8^2 \times 0{\cdot}4) + \\ + (16{\cdot}4^2 \times 0{\cdot}09) - (4{\cdot}176)^2 = 33{\cdot}69.$$

6. The variance of the sum of two random variables is given by the following:

$$\text{Var}(a + b) = \text{Var}(a) + \text{Var}(b) + 2\text{Cov}(ab).$$

Note that if $\text{Cov}(ab) = 0$ then $\text{Var}(a + b) = \text{Var}(a) + \text{Var}(b)$.
For example, referring to the example under '*expectations*' point 4:

a = revenue per ewe; b = cost per ewe.

$$\text{Var}(a) = 33{\cdot}69$$

$$\text{Var}(b) = (-5^2 \times 0{\cdot}01) + (-8^2 \times 0{\cdot}5) + (-12^2 \times 0{\cdot}4) + \\ + (-14^2 \times 0{\cdot}09) - (-10{\cdot}11)^2 = 5{\cdot}28.$$

$$\text{Cov}(ab) = (0 \times -5 \times 0{\cdot}01) + (9{\cdot}2 \times -8 \times 0{\cdot}5) + \\ + (18 \times -12 \times 0{\cdot}4) + (26{\cdot}4 \times -14 \times 0{\cdot}09) - (14{\cdot}176 \times -10{\cdot}11) \\ = -13{\cdot}14$$

$$\text{Var}(a + b) = (-5^2 \times 0{\cdot}01) + (1{\cdot}2^2 \times 0{\cdot}5) + (6^2 \times 0{\cdot}4) + \\ + (12{\cdot}4^2 \times 0{\cdot}09) - (4{\cdot}066)^2 = 12{\cdot}68.$$

Check:

$$\begin{aligned}\text{Var}(a + b) &= \text{Var}(a) + \text{Var}(b) + 2\text{Cov}(ab) \\ &= 33{\cdot}69 + 5{\cdot}28 - 26{\cdot}28 \\ &= 12{\cdot}69.\end{aligned}$$

Note in this case the covariance is negative so

$$\text{Var}(a + b) < \text{Var}(a) + \text{Var}(b).$$

7. If k is any constant, then

$$\text{Var}(k \,.\, a) = k^2 \,.\, \text{Var}(a).$$

For example, let a = gross margin per hectare of grain, which may take the values £20, £25, or £30, each with a probability of $\frac{1}{3}$ (see Table 4.6):

$$\text{Var}(a) = (20^2 \times \tfrac{1}{3}) + (25^2 \times \tfrac{1}{3}) + (30^2 \times \tfrac{1}{3}) - 25^2 = 16\tfrac{2}{3}.$$

For $k = 10$ hectares of grain, total gross margin $k \,.\, a$ will take on the values of £200, £250, or £300, again each with $P = \frac{1}{3}$:

$$\text{Var}(k \,.\, a) = (200^2 \times \tfrac{1}{3}) + (250^2 \times \tfrac{1}{3}) + (300^2 \times \tfrac{1}{3}) - 250^2 = 1666\tfrac{2}{3} \\ = 100 \times 16\tfrac{2}{3} = k^2 \,.\, \text{Var}(a).$$

8. If k_1 is one constant and k_2 another, then

$$\text{Var}(k_1 \,.\, a + k_2 \,.\, b) = {k_1}^2 \,.\, \text{Var}(a) + {k_2}^2 \,.\, \text{Var}(b) + 2k_1 \,.\, k_2 \,.\, \text{Cov}(ab).$$

This equation defines the variance of a combination of k_1 hectares of enterprise 1 and k_2 hectares of enterprise 2; it can be extended to cover any number of enterprises. The equation is a quadratic equation, so quadratic programming may be used to minimize this total variance subject to given fixed constraints.

9. In the special case when we are concerned with the proportion of the total area devoted to each crop, k_1 is equal to or less than 1 and $k_2 = 1 - k_1$. We may then write

$$\text{Var}(k_1 \,.\, a + (1 - k_1) \,.\, b) = {k_1}^2 \,.\, \text{Var}(a) + (1 - k_1)^2 \,.\, \text{Var}(b) + \\ + 2k_1 (1 - k_1) \,\text{Cov}(ab)$$

$$= {k_1}^2 \,.\, \text{Var}(a) + (1 - 2k_1 + {k_1}^2) \,\text{Var}(b) + (2k_1 - 2{k_1}^2) \,\text{Cov}(ab)$$

$$= \text{Var}(b) - 2k_1 \{\text{Var}(b) - \text{Cov}(ab)\} + {k_1}^2 \{\text{Var}(a) + \text{Var}(b) - \\ - 2\text{Cov}\,(ab)\}.$$

From this it may be shown that

(i) if $\text{Cov}(ab) < \text{Var}(a)$ and $\text{Var}(b)$,

variance is minimized by diversifying; and

(ii) if $\text{Cov}(ab) > \text{Var}(a)$ or $\text{Var}(b)$,

variance is minimized by concentrating on a single activity, the one with the smaller variance.[14]

[14] The variance of the combination $\text{Var}(c) = \text{Var}(k_1 a + (1 - k_1)b)$ is minimized where

$$\frac{d\text{Var}(c)}{dk_1} = -2(\text{Var}(b) - \text{Cov}(ab)) + 2k_1 \quad (\text{Var}(a) + \text{Var}(b) - 2\text{Cov}(ab)) = 0 \tag{4.3}$$

and

$$\frac{d^2\text{Var}(c)}{dk_1^2} = 2\{\text{Var}(a) + \text{Var}(b) - 2\text{Cov}(ab)\} > 0. \tag{4.4}$$

From eqn (4.3) we have

$$\text{Var}(b) - \text{Cov}(ab) = k_1\{\text{Var}(a) + \text{Var}(b) - 2\text{Cov}(ab)\}.$$

Hence

$$k_1 = \frac{\text{Var}(b) - \text{Cov}(ab)}{\text{Var}(a) + \text{Var}(b) - 2\text{Cov}(ab)}.$$

But from eqn (4.4) it follows that if $\text{Var}(a) \leqq \text{Cov}(ab)$ then $\text{Var}(b) > \text{Cov}(ab)$. Thus given that $0 \leqq k_1 \leqq 1$.

(i) if $\text{Var}(a) > \text{Cov}(ab)$ and $\text{Var}(b) > \text{Cov}(ab)$ then $0 < k_1 < 1$;
(ii) if $\text{Var}(a) > \text{Cov}(ab)$ and $\text{Var}(b) \leqq \text{Cov}(ab)$ then $k_1 = 0$;
(iii) if $\text{Var}(a) \leqq \text{Cov}(ab)$ and $\text{Var}(b) > \text{Cov}(ab)$ then $k_1 = 1$.

Suggestions for further reading

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Part II: Resource-use

5. Natural resources

Land as a factor of production

In classical economic theory (as expounded by Adam Smith, Ricardo, and Malthus) natural resources have a dominant role as the ultimate constraint on output and therefore as the cause of diminishing marginal returns to labour and capital. The 'original and indestructible powers of the soil' are distinguished from other factors of production in costing nothing to produce and in that the total quantity is fixed and cannot be augmented.

More recent theories of production, income distribution, and growth are based on the assumption that there are *no* fixed, non-augmentable factors of production. According to this view the area of land available for agriculture or other forms of production can be augmented at a cost by extending communications, or by clearing, irrigating, draining, fertilizing, and fencing the less fertile areas. Land is then just another form of capital and not an effective limit to economic growth.

However, it is apparent that in some densely populated countries the area of available land *is* an effective limitation on agricultural output. Furthermore there is increasing concern that for this planet earth as a whole there is an absolute limit to the non-renewable natural resources, which may soon be reached at current rates of world population growth.[1] Hence there is considerable justification for treating natural resources as distinct from capital and for discussing their particular characteristics and use. Although we shall concentrate on land in particular, most of the discussion in this chapter is relevant to other natural resources such as water.

The systems of land-use and more particularly the patterns of farming found in a particular area are influenced by the following features of the environment:

(1) density of agricultural population;
(2) technical knowledge and capital resources available;
(3) soil fertility;
(4) location in relation to markets, roads, and railways;
(5) land-tenure institutions.

[1] Meadows, D. H., Meadows, D. L., Randers, J., and Behrens, W. W. III (1972). *The limits to growth.* Earth Island, London.

These five features of the farmer's environment and their influence on farming systems will now be discussed in more detail.

Population density

To start with let us imagine 'that early and rude state of society which precedes both the accumulation of stock and the appropriation of land'.[2] This is somewhat unrealistic since even quite primitive subsistence cultivators must accumulate stocks of food, seed, and domesticated animals as well as simple implements. Nevertheless it is convenient to ignore capital and management at this stage and assume that land and labour are the only relevant factors of production. We further simplify the analysis at this stage by assuming that there is no variation in the quality of the land; it is homogeneous.

In a sparsely populated area, where there is still unused land, this resource is freely available to all. Labour is the only cost of production, and average product per person is maximized by using so much land that the marginal product per hectare falls to zero. Extensive systems of land-use, in which other resources are thinly spread over the productive area and the average product per hectare is low, are appropriate. As the farming population grows the area of land under cultivation, or used for grazing, is extended to keep pace so the number of hectares per person stays constant. The marginal product of the last hectare in use remains at zero.

Once all the available land is in use, further population growth forces producers to intensify, i.e. to use more labour to produce a larger average product per hectare. As a result the marginal product of the last hectare in use will be positive and will tend to rise as population density increases. In this way land becomes valued as a factor of production and land-tenure institutions are needed to establish rights of ownership and use.

This may be illustrated with the hypothetical data from Table 1.3 (p. 12), which gave the total product in bags of grain for various combinations of land and labour inputs. From these data we can derive the average product per labour unit, or per person, for various land–labour combinations as in Table 5.1. It is immediately obvious that the maximum average product per person is obtained when 1 unit of labour is applied to 7 hectares of land. When land is freely available this represents the economic optimum level of land-use since the marginal product per

[2] Smith, A. (1776). *The wealth of nations* (ed. E. Cannan (1904)), Book I, Chapter VII. Methuen, London.

hectare is zero. This land–labour combination is shown at point A in the isoquant diagram, Fig. 5.1, and the response curve, Fig. 5.2, both of which are drawn from the tabulated data. Clearly it would be irrational to

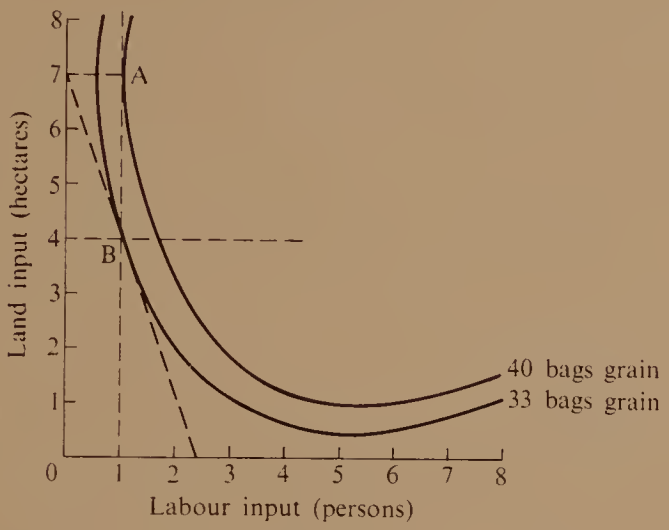


FIG. 5.1. The effect of changing land : labour ratios on the marginal products. So long as land is freely available, labour productivity is maximized on the upper ridge line, i.e. where marginal product per hectare of land is zero (point A). Once all land is in use and the ratio of land to labour shifts downwards, land yields a positive marginal product or rent (point B).

TABLE 5.1

Average product per person in bags of grain

		Labour input (persons)						
		1	2	3	4	5	6	7
Land input (hectares)	1	7	10	$10\frac{1}{3}$	$9\frac{1}{4}$	8	$6\frac{1}{2}$	5
	2	18	$16\frac{1}{2}$	15	13	$11\frac{2}{5}$	$9\frac{1}{2}$	$7\frac{5}{7}$
	3	27	21	$18\frac{1}{3}$	$15\frac{1}{2}$	13	$11\frac{1}{6}$	$9\frac{2}{7}$
	4	33	24	$20\frac{1}{3}$	$16\frac{1}{2}$	$14\frac{1}{5}$	$12\frac{1}{6}$	$10\frac{2}{7}$
	5	37	$26\frac{1}{2}$	22	$18\frac{1}{3}$	$15\frac{1}{5}$	13	11
	6	39	29	23	19	$15\frac{4}{5}$	$13\frac{1}{2}$	$11\frac{3}{7}$
	7	40	30	$23\frac{1}{3}$	$19\frac{1}{4}$	16	$13\frac{2}{3}$	$11\frac{4}{7}$
	8	39	30	$23\frac{1}{3}$	$19\frac{1}{2}$	$16\frac{2}{5}$	14	12

extend the area of land in use beyond this level since the marginal product of the eighth hectare per person is negative. However, as population grows, each additional person may claim an additional 7 hectares of land from the unused territory to produce an extra 40 bags of grain. The

marginal product per hectare will remain at zero and the marginal and average product per person will stay constant at 40 bags of grain.

When all the available land is in use, further population growth must cause a decline in the number of hectares per person. For instance, if the increase in population is such as to reduce the amount of land per person

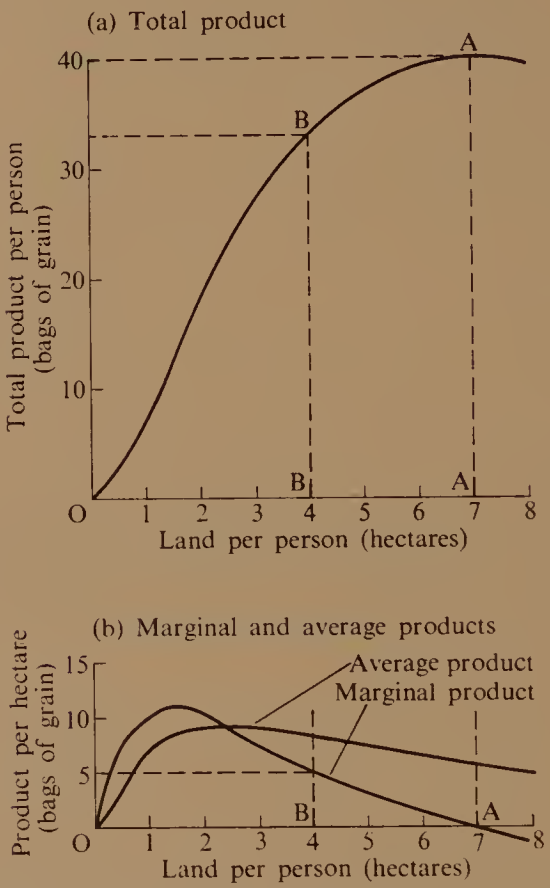


FIG. 5.2. The effects of varying land use per person. The response curve is the vertical section of the response surface for one unit of labour input, whereas Fig. 5.1 was a plan view. Again we see how, as the amount of land available falls from 7 hectares to 4 hectares per person (a decline from OA to OB) the marginal product (rent) per hectare rises from zero to 5 bags of grain.

from 7 to 4 hectares (from point A to point B in the diagrams), the average product per person must fall from 40 bags to 33 bags of grain, while the marginal product per hectare of land rises from zero to 5 bags of grain. It would be profitable for a farmer to hire an additional hectare of land for anything up to 5 bags of grain per hectare if this were possible. If such a hire charge is paid for the use of land it is called '*rent*'.

Note that in developing this argument we assume that the optimum *scale* of operation is selected. This example is simplified in that the optimum scale of operation appears to be a 1 man unit throughout. From Table 5.1 it is clear that a combination of 1 person and 7 hectares of land yields a higher average product per person than any other combination of the two inputs. When the available land per person is restricted to 4 hectares the 1 man unit still yields the highest average product per person at 33 bags of grain. Thus 2 persons on 8 hectares produce only 30 bags of grain per person. In the real world the optimal size of labour force is likely to vary with the amount of land farmed.

We have also assumed that farmers are profit maximizers, profit in this case being measured in bags of grain. Now, in the first situation, where surplus land is freely available, an individual farmer might well be satisfied with less than 40 bags of grain per year. This need not alter the theory fundamentally. All it means is that he could satisfy his wants by working fewer hours than we had assumed in our analysis. However, he would still maximize the average product per hour of work (or minimize the effort needed to produce his grain requirements) by using land up to the level where the marginal product per hectare is zero. Once all the land is in use, and if there is no change in technology, the average product per hour of work must fall and the marginal product per hectare of land must rise. Land then can command a rent.

To the classical economist, David Ricardo, rent appeared to be a pure surplus, since the land would be available even if no rent were paid. This conclusion follows from two assumptions: (1) that the total area available is fixed, and in no way dependent upon the return obtained and (2) that land has no alternative uses other than grain production. The first assumption implies that there is no direct cost of supply, in the sense that a subsistence level wage may be viewed as the direct cost of labour supply, while the second assumption implies that there is no opportunity cost. For most land-use decisions these assumptions are misleading. In so far as clearing, drainage, provision of roads, and fencing may be needed to make usable land available to producers, and conservation practices may be needed to maintain it in a usable condition, the total area of available land will be influenced by the return obtained. Furthermore land invariably has alternative uses: farming competes with forestry, recreational, or urban uses; grassland farming competes with arable farming; grains compete with roots; and one farmer competes with another farmer for the use of a particular plot of land. Hence each hectare of land has an opportunity cost. Even if we assume in our hypothetical example that land cannot be hired, the transfer of one hectare

to some other use would nevertheless cost the farmer, and society as a whole, 5 bags of grain, since this amount would be forgone.

In practice it may not be possible to substitute labour for land as is implied by the smooth curved isoquants of Fig. 5.1. The linear, fixed-factor-proportions model of Chapter 2 may be more appropriate. In this case it will be recalled that, so long as the amount of land available is *not* an effective constraint on production, the marginal product per hectare is zero. Once the land area is an effective limitation on output the marginal product per hectare must rise. These conclusions are identical with those we have already reached, but if we now assume that there is only one known technique of production, which means there is only one process it is not possible to vary the proportions in which land and labour are used.

Let us assume that the existing process requires 7 hectares of land per person and yields 6 bags of grain per hectare, or 42 bags per person. So long as there are more than 7 hectares of land available to each worker the marginal product per hectare is zero and some land will remain in disposal. Once all the land is in use, however, and the area available falls below 7 hectares per person labour *must* be displaced. If there is no alternative productive work to do, unemployment or under-employment must result—i.e. there will be labour in disposal, the marginal product per person or per hour of work falls to zero, and the marginal product per hectare of land rises to 6 bags of grain. The main difference from the earlier discussion is that now labour is displaced, rather than substituted for land. Furthermore, any attempt to increase the intensity of land-use in such circumstances may result in a breakdown of the system, a decline in the fertility of the soil, and ultimately erosion.

Many instances have been documented, from parts of Africa, where, under increased population pressure, the system of shifting cultivation has broken down and areas of land have become devastated.[3] Similarly, overgrazing of communally held pasture lands has lead to destruction of grass cover and finally to soil erosion. In Latin America and the southern states of North America, large areas of topsoil have been eroded following intensive land-use. However, in many other parts of the world, increasing numbers of farmers and increasing intensity of land-use have been accommodated with no deterioration of soil fertility, no decline in the average and marginal product per person, and no increase in rents as a proportion of total product. This has been possible as a result of technological change.

[3] Allen, W. (1965). *The African husbandman.* Oliver and Boyd, London.

Technological change

Technology is the pool of productive activities currently used by a particular society. It determines what can be produced with given amounts of resources. Technological changes include the introduction of new products, or product innovations, and new methods of production, called process innovations. We are mainly concerned with the latter here. More precisely a process innovation is defined as any *adopted* improvement in method of production which reduces average costs per unit of

TABLE 5.2

Technological change with fixed-factor-proportions activities

	Original activity	Neutral innovation	Labour-saving innovation	Land-saving innovation
Product per unit (bags)	42	63	63	42
Labour per unit (persons)	1	1	1	1
Land per unit (hectares)	7	7	9	6
Average product (per hectare)	6	9	7	7
Percentage increase in average product over original activity (per hectare)		50	17	17
Percentage increase in average product over original activity (per person)		50	50	0

output even if input prices remain unchanged. Alternatively this implies that the average productivity of at least one resource is increased.

Process innovations are further classified according to whether they are neutral and affect all resources equally, or biased in the direction of saving one resource more than another. Thus the introduction of irrigation and chemical fertilizers which increase yield per hectare are likely to be land-saving in effect, while mechanization may save labour. Where the existing technology consists of a single fixed-factor-proportions activity, the distinction is clear. Thus let us assume that this 'original activity' requires 7 hectares of land per person and yields 42 bags of grain, as shown in the first column of Table 5.2. A new activity which requires the same amount of land per person but which produces a yield

of 63 bags of grain is neutral as between land and labour. It simply increases the productivity of both in the same proportion, 50 per cent, as shown in the second column. The activity shown in the third column allows each person to cultivate a bigger area to produce a larger product than the original activity. It is therefore labour-saving. The last column shows a land-saving activity, since the original output per person is obtained from a smaller area of land.

The situation is more complicated when factor substitution is possible within the existing technology. As shown in the previous section, even in the absence of technological change, labour inputs per hectare may be increased with a resultant rise in the average and marginal product per hectare and a decline in the average and marginal product per person. This simply represents the substitution of one factor, labour, for another factor, land, using the same technology, and may be called a change in technique. A technological change differs from a change in technique in that it represents a shift in the production function: a given combination of inputs yields a larger total product. Although technological innovations may lead to changes in the combination of resource inputs, it is convenient to classify them according to their effects on the marginal products of resources assuming *no change* in the combination used. Now, any technological innovation necessarily raises the average productivity of each resource in the same proportion as it raises total product. For instance, if the output obtained from a given combination of resources is increased by 10 per cent, it follows that the average product per unit of each resource must be raised by 10 per cent. However, the marginal product per unit of each resource may not be raised by the same percentage. In fact we may now define a neutral innovation as one which increases the average product and the marginal product of each resource in the same proportion. A land-saving innovation, on the other hand, does not increase the marginal product per hectare to the same extent as it raises the average product for a given combination of inputs. This may be interpreted as a fall in the elasticity of response to land, since this elasticity is the ratio of the marginal product to the average product. Similarly a labour-saving innovation is defined as one which reduces the elasticity of response to labour.

It is important to distinguish between neutral and biased innovations because the latter, unlike the former, alter the distribution of the total product among the factors of production. This may be illustrated with the examples given in Table 5.3 and Figs 5.3 and 5.4 which are simplified by assuming (1) that there are only 2 factors involved, namely land and labour; (2) that there are constant returns to scale so that the returns to

the two factors account for the total product exactly (see Chapter 3); and (3) that the land-saving innovation brings about no change in the marginal product of land. Nevertheless, these examples serve to illustrate the general rule that for a fixed combination of inputs a land-saving innovation reduces the opportunity cost or shadow-rent of land as a proportion of the total product. This follows directly from our definition of a land-saving innovation.

Alternatively, we may now relax our assumption that the combination of inputs is fixed and consider the influence of innovations on the quantities of different resources, including land, likely to be hired. For a single

TABLE 5.3

Technological change with variable-factor-proportions

	Original technology	Neutral innovation		Land-saving innovation	
	(bags of grain)	(bags of grain)	(percentage increase (approx.))	(bags of grain)	(percentage increase (approx.))
Total product = average product per person (from 4 hectares)	33	40	21	40	21
Average product per hectare	8·25	10	21	10	21
Marginal product per hectare, 'rent'	5	6	21	5	0
Total 'rent'	20	24	21	20	0
Marginal product per person = total product − total 'rent'	13	16	21	20	54
Total 'rent' as proportion of total product (per cent)	60	60		50	

individual we may assume that all prices are constant; they will not be affected by his adoption of new technology. Thus if the marginal product per unit of any factor of production increases as a result of an innovation the economic optimum rate of use of that factor will be increased; the farmer will have an incentive to use more of that factor. But our definition of a land-saving innovation implies that the marginal product per hectare of land is not raised by as large a proportion as the marginal products per unit of other inputs. Indeed, the marginal product per hectare may not be raised at all. Thus it seems likely that a land-saving innovation will provide a farmer with an incentive to increase inputs of other resources in relation to the area of land he uses: i.e. to intensify. However, unlike the situation where there is no innovation, increased

intensity of land-use is not accompanied by a corresponding increase in the marginal product per hectare.

If the innovation is adopted by large numbers of producers it is likely that prices will be affected. As output is increased there will be a tendency for the product price to fall, so that consumers will be persuaded to buy the extra output, unless consumer demand is rising for some other reason. Similarly, increased use of resources by many grain farmers may

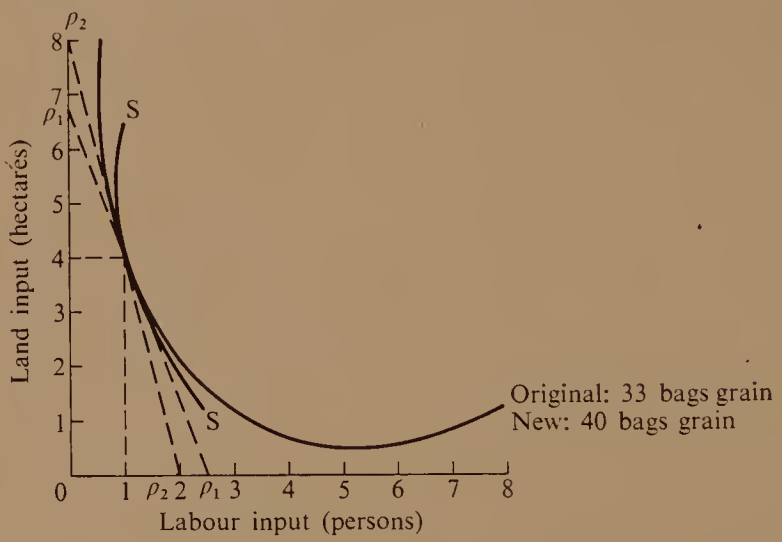


FIG. 5.3. The classification of process innovations. A neutral innovation simply causes a vertical shift of the whole production surface. Thus the 33-bag isoquant for the original technology may become, say, the 40-bag isoquant after a neutral innovation. However, a land-saving innovation changes the slope of the isoquant to that of SS; the rate of technical substitution of labour for land increases. As a result the 'price' of land falls in relation to the price of labour for a given land:labour ratio from the slope of $\rho_1\rho_1$ to that of $\rho_2\rho_2$.

cause input prices to rise. However, for a land-saving innovation the rise in land price is likely to be proportionately less than the rise in prices of other inputs. Relatively speaking, rents will tend to fall. In this way land-saving innovations may counteract the effects of increasing intensity of land-use, which would otherwise cause a relative rise in rents.

The assumption that land and labour are the only two factors of production is clearly unrealistic. Capital and management are necessary for any productive activity to take place. Indeed all innovations are 'embodied' in new forms of capital or 'new sources of income streams'. However, if we now admit that these other factors are involved, our definitions of neutral and land-saving innovations are still valid. So too

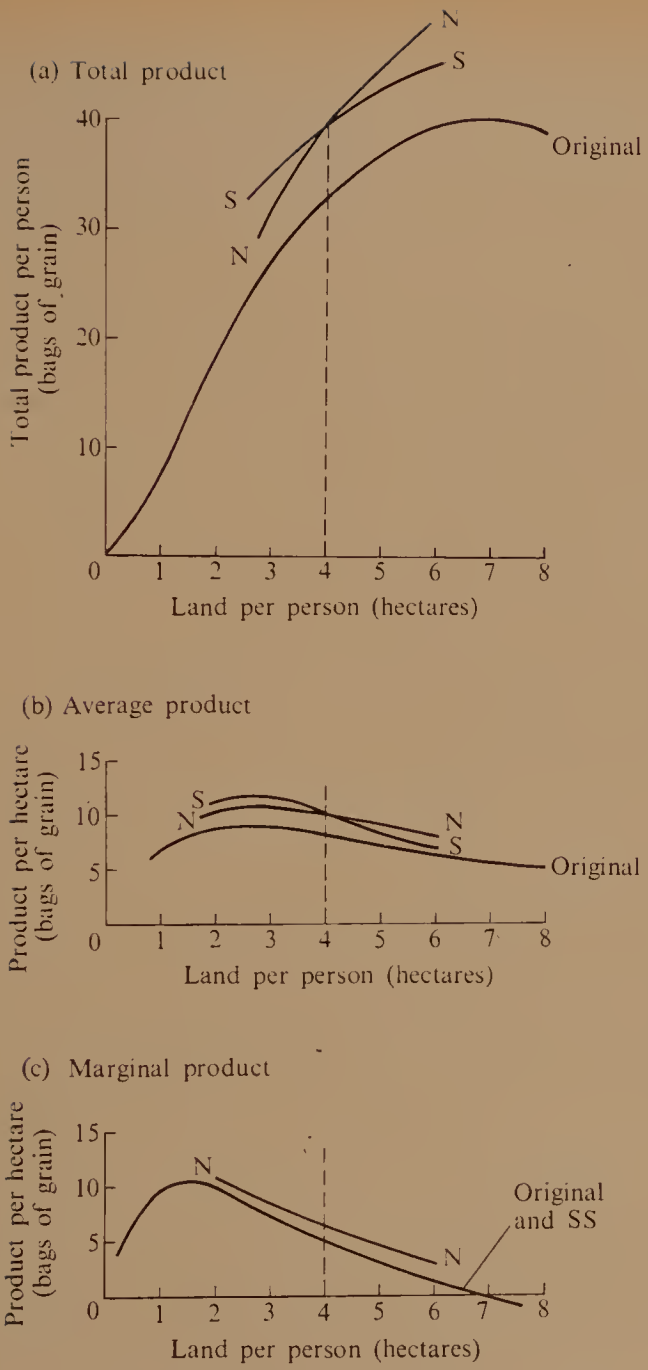


FIG. 5.4. The effects of innovation on productivity of land. For a given land:labour ratio (in this case 4 hectares per person) a neutral innovation increases both average and marginal product of land by an equal amount (line NN in each diagram), whereas, a land-saving innovation increases average product by a larger amount. Indeed, marginal product may remain constant, as in this illustration (line SS in each diagram).

is the argument, set out above, that land-saving innovations may be expected to cause rents to fall in relation to total product and the prices of other factors. Sir John Hicks has recently argued ((1973): in his *Capital and time*, p. 123, Clarendon Press, Oxford) that 'induced invention, though it may very probably cause a fall in rents in the short run, must in the longer run raise them'. However, if population growth can be controlled so that labour becomes the scarce factor, some of the gains from progress, even the long-run gains, can pass from land to labour. Thus our conclusions may be applied in the high-income countries of Europe and America, where the population or labour force engaged in agriculture is actually declining but where the intensity of land-use is still rising as a result of increased inputs of capital per hectare. In these countries during this century, land rents per hectare have risen surprisingly little and, relative to other factor prices, have actually declined. This reflects the land-saving bias of most agricultural innovations.

Many agricultural innovations are clearly land-saving in effect. The change from shifting cultivation using fire and digging-stick to settled rotational bush-fallowing with hoe cultivation is land-saving, as is the change from the latter system to continuous annual cropping based on use of the plough. The introduction of chemical fertilizers, pesticides, new seed varieties, and irrigation, all of which increase crop yields per hectare, are likely to save land. Similarly, intensive livestock production involving restricted grazing or permanent housing and feeding by hand or machine may be expected to raise average product per hectare without increasing marginal product to the same extent.

The apparent land-saving bias in agricultural innovations may be attributed to various causes. First, individual farmers may be induced to seek out land-saving innovations in order to avoid diminishing marginal returns to other factors as intensity of land-use is increased. Now it is arguable that, if land and other inputs may be hired at prices over which the individual farmer has no control, there is no incentive or mechanism to induce 'appropriate' factor-saving innovations. The farmer's aim is to reduce total costs irrespective of whether the saving is made on payments for land, labour, or capital. However, in reality it is rarely possible to hire additional land in just the quantities and at just the time it is wanted. A farmer with a given area of land has an incentive to intensify and make land-saving innovations.

Secondly, the responsibility for financing most agricultural research lies with national governments. Most farm businesses are too small to sponsor research on an individual basis, and the benefits obtained when research findings are introduced as innovations are freely available to all.

This service is provided to farmers because most governments for one reason or another consider that it is in the national interest to raise the productivity of resources employed in agriculture. Governments nowadays are certainly conscious of the absolute limitations on the total area of land available for agriculture globally, even if individual farmers are not influenced by it. Thus research into new techniques which are likely to save land will be encouraged.

Finally, most agricultural research workers are generally concerned with biological or technical efficiency; with maximizing physical growth and reproduction rates. For most crops the unit of production is an area of land, and most research is concerned with raising yields per hectare. Thus there may be a land-saving bias in agricultural research. Naturally, farmers will adopt research results which reduce their costs and which are acceptable in other ways, regardless of whether they have a land-saving bias.

Soils and climate

So far we have assumed that land is homogeneous or all of one uniform type. This is clearly not the case; the total product per hectare which can be obtained at a given rate of labour and capital input is influenced by the natural environment, the inherent fertility of the soil, the topography, and climate. Thus in a sparsely populated region we would expect the more fertile land with the more equable climate to be cultivated first. Once all this land is in use, further population growth will cause it to be cultivated more intensively but, before this intensification has proceeded very far, the returns per unit of labour and associated capital will have fallen to a level which could be achieved by cultivating land of poorer quality. Thus total returns are maximized by extending production onto areas of hitherto unused poor land rather than by intensifying production on good land.

This concept is illustrated in Fig. 5.5, which shows the effect of varying the area of land used per person on the total product per person for land of two different qualities. The lower diagram shows the corresponding variation in the marginal product per hectare on the two types of land. The maximum total product per person which can be obtained is shown to be higher on good land then on poor land so if there is sufficient good land available it would be irrational to cultivate the poor land. However, once the area of good land available falls to 3·5 hectares per person (point A), the same total product of 30 bags of grain per person can be obtained by cultivating 10 hectares of poor land (point C). It should be noted that a larger area of poor land is needed to produce a given total

product per person. Hence the poor land is farmed less intensively, in larger units, and supports a lower population density. So long as there is sufficient poor land available, labour productivity is maximized by extending cultivation to point C, where marginal product is zero, as shown in the lower diagram. Yet because the good land is farmed more intensively with only 3·5 hectares per person it has a positive marginal product, in this case about 6 bags of grain per hectare.

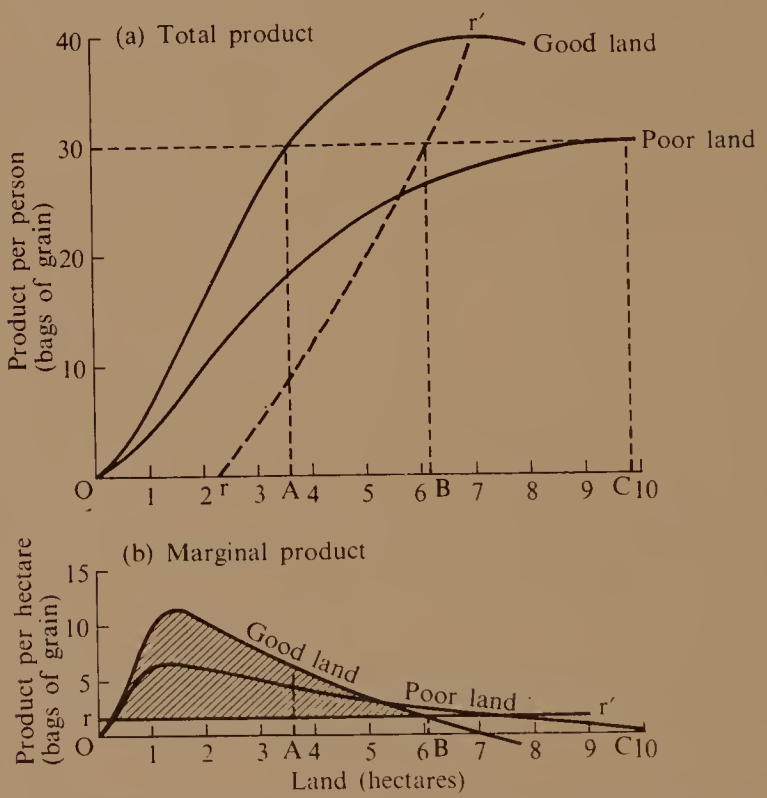


FIG. 5.5. Extending production onto poor land. Once the area of good land per person falls to OA hectares then the same total product per person can be earned by extending cultivation onto poor land at an intensity of OC hectares per person. If rent is actually paid then production will be extended onto poor land when there are OB hectares of good land per person, which in this case implies a rent of 1½ bags of grain per hectare of good land.

If the marginal product per hectare was actually paid to a landlord as rent, the net returns per person would fall more rapidly with reduction in the area of good land available, as shown by the broken line rr′. This line was constructed by subtracting the total rent, which is the number of hectares times the marginal product per hectare, from the total product for each level of land-use. Now, the net return per person falls to 30 bags

on 6 hectares of good land (point B) at which point there will be an incentive to extend cultivation onto poor land. For 6 hectares of good land the marginal product per hectare is only $1\frac{1}{2}$ bags. In the lower diagram, the shaded area above this rent line for good land is equal to the total area under the marginal product curve for poor land, since these

TABLE 5.4

Linear programming with land of different qualities

(a) The problem

	Constraint level	Grain production on good land	Grain production on poor land
Product of grain per unit (bags)		6	3
Labour (persons)	3	$\frac{1}{7}$	$\frac{1}{10}$
Good land (hectares)	14	1	0
Poor land (hectares)	14	0	1

(b) The solution

			Bags of grain
Grain production on good land; 14 hectares yields	14×6	=	84
Grain production on poor land; 10 hectares yields	10×3	=	30
(Poor land disposal = 4 hectares)			
	Total		114
Shadow-prices			
Labour at 30 bags of grain per person costs	3×30	=	90
Good land at $1\frac{5}{7}$ bags of grain per hectare costs	$14 \times 1\frac{5}{7}$	=	24
Poor land: zero per hectare			0
	Total		114

areas represent the total net return per person on each type of land. Once all the poor land is in use, further population growth or further increases in output will depend upon both types of land being used more intensively so that their marginal products per hectare will rise. Thus the marginal product per hectare of good land might again rise to 6 bags of grain and that of poor land would then be around $4\frac{1}{2}$ bags of grain. At the same time, if it were possible, production would be extended onto yet poorer quality soils.

The same concepts may be illustrated by a simple linear programming exercise such as that presented in Table 5.4 and Fig. 5.6. Here again less labour is used per hectare of poor land than is used per hectare of good land: i.e. poor land is farmed less intensively. In the optimum solution

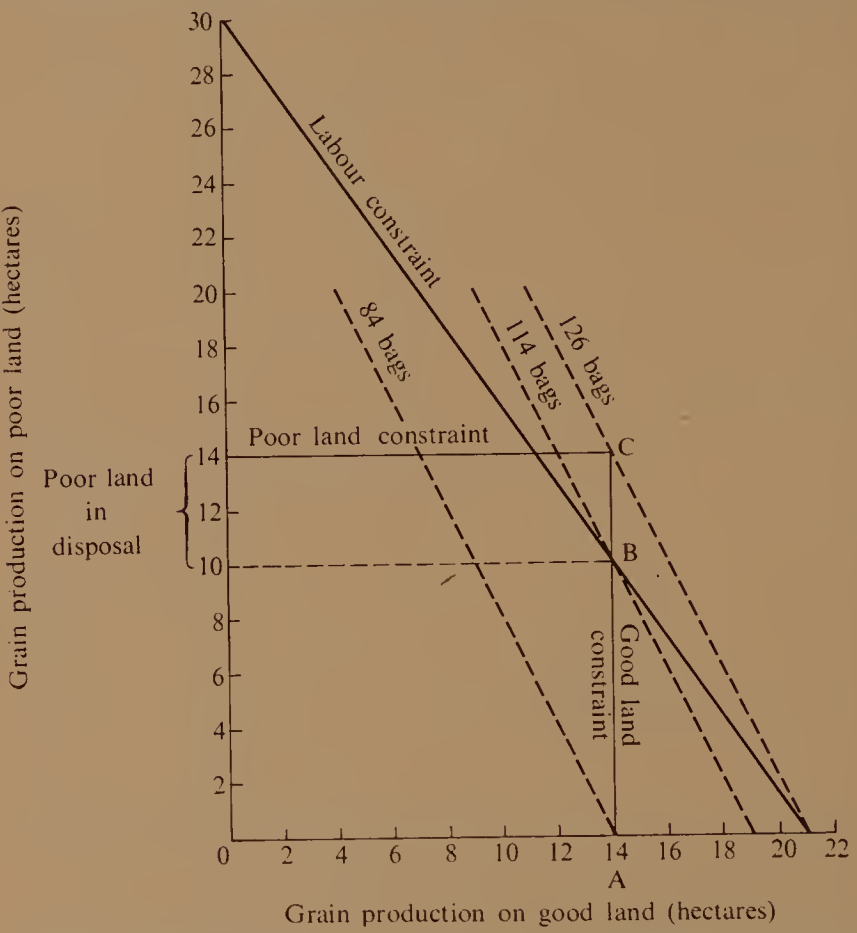


FIG. 5.6. Extending production onto poor land; the linear model. With only 2 persons cultivating 14 hectares of good land (7 hectares per person) 84 bags of grain would be produced and no poor land would be cultivated (point A). As the labour force increases, poor land is brought into cultivation. With 3 persons employed 10 hectares of poor land are cultivated to produce a total of 114 bags of grain (point B). With more than 3·4 persons employed, poor land becomes an effective constraint and 126 bags of grain is the maximum that can be produced (point C).

the marginal product per hectare of good land is $1\frac{5}{7}$ bag of grain while the marginal product per hectare of poor land is zero.

Note however that if there were less than 2 persons available to farm the given areas of land no poor land would be used and the marginal product per hectare of good land would be zero. Alternatively with more

than 3·4 persons employed both types of land would be effective constraints, the maximum total product would be 126 bags of grain, and the marginal products per hectare of the two types of land would be 6 bags for good land and 3 bags for poor land.

In fact, there is an almost continuous range of variation in soil fertility even between the plots of land cultivated by an individual farmer. In order to maximize profit from land of variable quality, farmers should

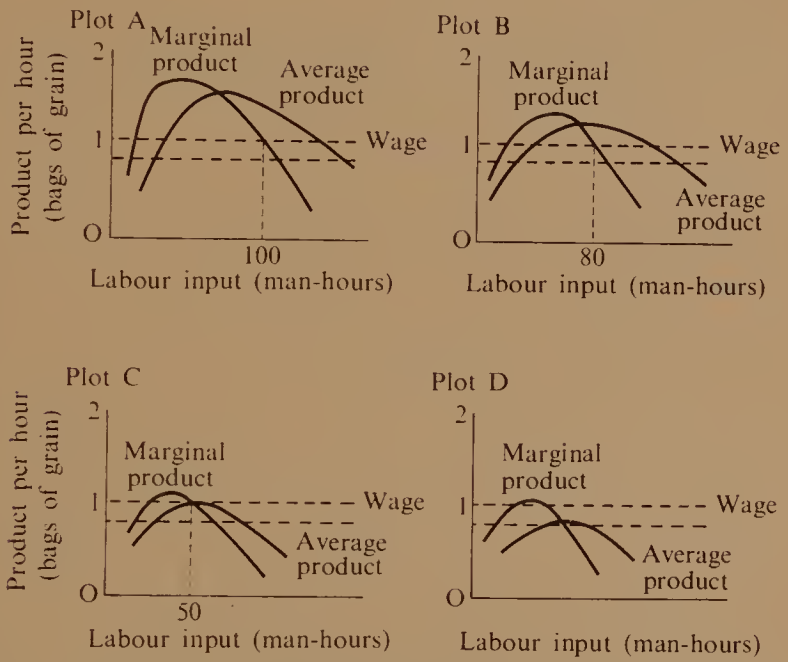


FIG. 5.7. Land quality and intensity of use. The economic optimum rate of labour-use per hectare is higher on the most fertile plot, A, than on the other plots of poorer quality, i.e. plot A is farmed more intensively. Plot D is sub-marginal at a wage per man-hour of 1 bag of grain, and will not be brought into use unless the wage falls to 0·8 bags (the lower dashed line).

use the principle already given in Chapter 3; labour and other inputs should be allocated between plots so that the marginal return is equal from all plots, as shown in Fig. 5.7. These four plots are assumed to be equal in area but of varying fertility, ranging from plot A, the most fertile, to plot D, the least. If labour is allocated so that the marginal product per hour is the same on each plot, then plot A is farmed more intensively than plot B which in turn is farmed more intensively than plot C. In this hypothetical example, if the wage per hour of labour is 1 bag of grain, then plot D should not be cultivated at all because the average product is less than the cost of labour, while it only just pays to

cultivate plot C if there are no costs other than labour, since the *average* product of labour just equals the wage rate. Only a slight increase in labour costs, or a fall in productivity or product value would cause plot C to go out of cultivation. It is therefore known as 'marginal land' since it would be the first to go out of production under adverse conditions. Plot D is referred to as sub-marginal.

Agricultural production can be increased in two ways, either by intensifying production on land which is already farmed or by extending production onto land which previously was not farmed. If because of farming population growth or for some other reason the value of food products rises in relation to the value of labour, there will be a tendency for both intensification and extension to occur. In Fig. 5.7 this new factor–product price ratio is represented by the lower horizontal dashed line. The new economic optimum occurs at a higher level of labour input on plots A, B, and C so production is intensified, but now if labour is the only cost then plot D is no longer sub-marginal so production may be extended onto this plot. Conversely, where agricultural growth and development is occurring the value of labour may rise in relation to the value of food products, and this may result in land which was previously farmed becoming sub-marginal. This has occurred in the past on hill-lands in Britain and on the desert fringes in Africa.

Soils and climates also influence the choice of crop and livestock enterprises. Land which is sub-marginal for one product may be super-marginal for another; for instance land suited to sheep grazing may be quite unsuitable for cereals or vegetable growing. Furthermore, some enterprises represent a more intensive use of land than others. Intensive crops are those like vegetables that require large inputs of labour, fertilizer, and possibly irrigation water per hectare. Cereal growing is generally a somewhat less intensive form of land-use and livestock grazing on permanent pastures probably the least intensive of all. In fact deep, well-watered, and well-drained alluvial soils are usually suitable for a wide range of products, whereas cold, infertile hill-lands or the arid lands of the tropics may not be suited to any agricultural use other than extensive stock grazing. Farmers in the more fertile areas have the widest range of choice and, in most enterprises, an absolute advantage over farmers of poorer soils. However, this does not mean that production of most products should be restricted to the more fertile areas. In fact everyone will be better off if each farmer concentrates on those enterprises for which he has the greatest *comparative* advantage (or least comparative disadvantage).

Let us imagine just two farmers, farmer A on good land (farm A)

and farmer B on poor land (farm B), both producing the two crops roots and grain. For further simplification suppose that labour, capital, and management resources cannot be transferred from one farm to the other but are readily moved from one crop to the other on each farm. Also suppose initially that each farmer devotes half his land to one crop and half to the other. The marginal product of the last hectare of land devoted to each crop on each farm is given in Table 5.5. It will be noted

TABLE 5.5

Comparative advantage in grain and root production

	Marginal product per hectare in grain production (bags)	Marginal product per hectare in root production (tonnes)
Farm A: good land	6	2
Farm B: poor land	4	1
	Valued at £1 per bag (£)	Valued at £3.50 per tonne (£)
Farm A: good land	6	7
Farm B: poor land	4	3.50

that the marginal product per hectare of land is higher for both crops on farm A, which therefore has an absolute advantage over farm B. However, the ratio of the marginal products (i.e. the rate of product transformation) differs on the two farms, which means that the total production of both crops may be increased if there is more specialization and trading of products.

Thus suppose that farmer B obtains 1 tonne of roots from farmer A. This means that farmer B can devote one less hectare of his land to root production and still have as large a quantity of roots as before. If he now uses this hectare of land, which is released, in grain production he will produce an additional 4 bags. Farmer A, on the other hand, in order to make good the tonne of roots received by farmer B must transfer $\frac{1}{2}$ hectare of his good land from grain to root production. His grain production will then be reduced by $6/2 = 3$ bags of grain. In the circumstances, if farmer B gives farmer A somewhere between 3 and 4 bags of grain in exchange for the tonne of roots both farmers will be better off. While both are using the same amount of land and other resources and both have the

same amount of roots as before they each have more grain than they had previously. It will be noted that these mutual benefits of increased specialization and trading arise because farmer A has a comparative advantage in root production.

Now consider the money values of the marginal products given in the lower part of Table 5.5. At the prices given, 1 tonne of roots exchanges for 3·5, that is between 3 and 4 bags, of grain. The conclusion that it is profitable for farmer A to use more land for roots and for farmer B to use more land for grain now clearly follows from the rule for profit maximization given in Chapter 3, namely that each unit of land, or other input, should be used where it will add most revenue. The economic optimum allocation is found when the value of the marginal product is equal in all uses. As we have seen this need not imply complete specialization if there are supplementary or complementary relationships between enterprises. Furthermore farmers who are risk averse may diversify to reduce their income variability. Nevertheless differences in comparative advantage do account for differences in farming systems.

Although the above discussion is concerned with comparative advantage as between different farms it may equally well be applied on the one hand to international trade or, on the other to different fields of differing soil type within a single farm.

Location

By its very nature, agricultural production is widely dispersed over the available land area, which means that, in all but purely subsistence societies, a correspondingly widespread transport network is needed to take produce from the farms to the market places in the towns and cities and to bring supplies and equipment to the farms. Now it may be that the establishment of urban centres and the transport network is itself dependent upon the location of agricultural production. Nevertheless the existence of towns and the system of communications is due to decisions made in the past and is certainly outside the control of individual farmers. Hence for our purposes it is reasonable to assume that these are predetermined, fixed features of the farmer's environment. Our purpose now is to consider how the location of farms in relation to these features influences the pattern of land-use. Incidentally, this analysis, unlike that based on variations in soils and climate is relevant to the location of any kind of economic activity not just agriculture.

For simplicity, let us consider a single isolated market centre, supplied with a given agricultural product, say grain, from farms in the surrounding region. Now the net price per tonne of grain, received by a

particular farmer will be the prevailing market price minus the transfer cost per tonne, which is the cost of getting the grain from the farm to the market centre. Part of the transfer cost is incurred in assembling, loading, and unloading and is independent of the distance to the market. This is known as the terminal cost. However, the actual cost of transport is likely to vary with distance, though not necessarily directly. In general the transport rate per kilometre is lower for long hauls. Hence, because of the fixed terminal cost element and falling transport rates, we may assume that the total transfer cost per tonne of grain increases with distance but at a decreasing rate.

On the basis of these assumptions we may draw a hypothetical product price gradient, which relates farm price to distance from the market centre as shown in Fig. 5.8(a). We assume that a single uniform market price OP is established for this crop at the market centre. However, the price received by the farmer is net of transfer cost, hence, as the transfer cost increases with distance from the market, the residual farm price decreases. It finally falls to zero at distance OD from the market. Even on farms immediately adjacent to the market centre the price is reduced by the terminal cost AP. The differences in farm price as between different locations will influence the pattern of production.

As the farm price declines with distance from the market centre so too must the value of the marginal product for all inputs. This is so because the value of the marginal product equals the marginal physical product multiplied by the price.[4] Thus for a variable input such as labour, the whole marginal product curve, measured in value terms, will shift downwards. As a result the optimal intensity of land-use declines with increasing distance from the market centre. This effect may be illustrated with the aid of Figs 5.5, 5.6, and 5.7 if 'land quality' is now assumed to be solely dependent on distance from the market centre, 'good land' being land near the market centre and 'poor land' being more distant. Thus land near the market centre is farmed more intensively in smaller units of area, and supports a higher population density at a given level of money income per head than more remote areas. In Fig. 5.7, plot D is now assumed to be sub-marginal because of its remoteness from the market centre rather than because of its low fertility.

At a given intensity of land-use the value of the marginal product per

[4] Alternatively this effect may be expressed by saying that as the product price declines the *real* unit factor cost (measured in units of the product) must increase. This is quite apart from the possibility that the unit factor cost may be raised by the cost of transferring inputs from the market centre to the farm. This latter possibility is ignored in the above analysis, though it could easily be incorporated.

hectare of land would decline with increasing distance from the market centre simply as a result of falling product price. However, as the intensity of land-use falls so too does the marginal product per hectare. Thus for two reasons: (1) the direct effect of falling product price, and

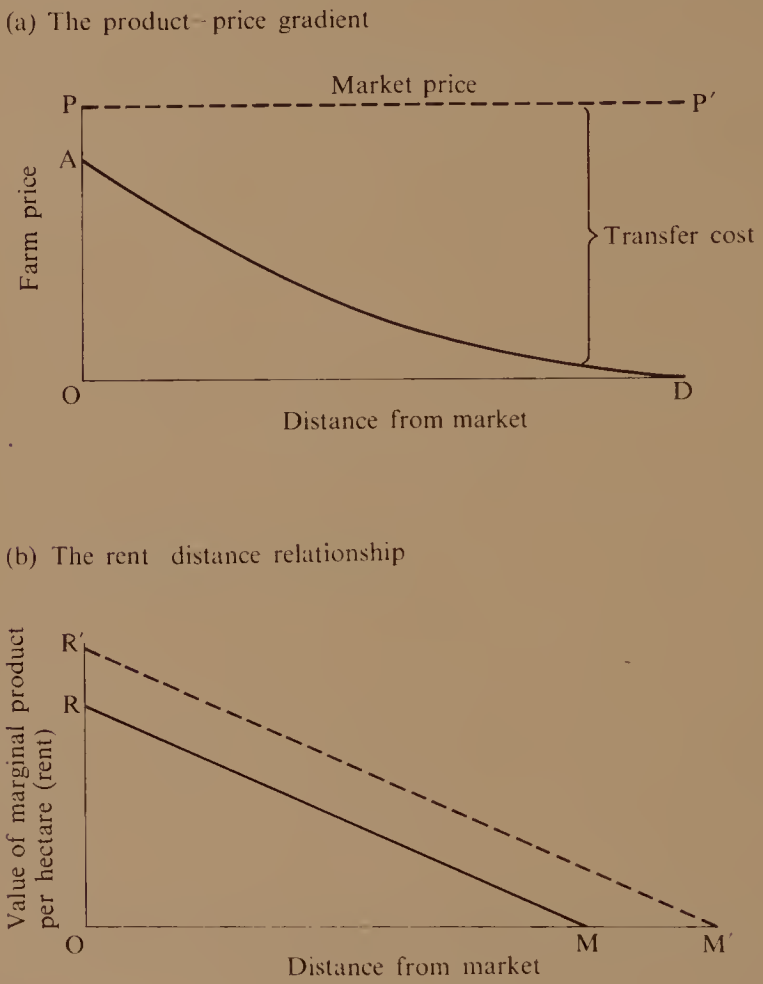


FIG. 5.8. Location, prices, and rents. (a) Since the transfer cost increases with distance from the market, the net price received by farmers decreases with distance. (b) As a result land rents also decline with distance from the market until, at distance OM, land is marginal and commands zero rent. An increase in productivity or prices as shown by the broken line raises rents by RR′ and extends the margin of cultivation by MM′.

(2) the indirect effect through declining intensity of land-use, the value of the marginal product per hectare (i.e. the rent) is likely to fall as distance from the market centre increases. We may therefore predict a rent–distance relationship as shown in Fig. 5.8(b). It is drawn as a straight line

for simplicity, though it could well take some other form. The decline is likely to be more rapid than the decline in price with increasing distance because of the additional effect of falling intensity. This figure implies that land will be farmed with decreasing intensity out to a distance OM from the market centre. At this distance land is marginal and beyond it land is sub-marginal for grain production. This diagram may be used to show the probable effects of a rise in the price of grain in relation to the prices of variable inputs. For reasons given above such a price rise will provide an incentive for more intensive use of grain land, thus both as a direct result of the price rise and as a result of increased intensity of

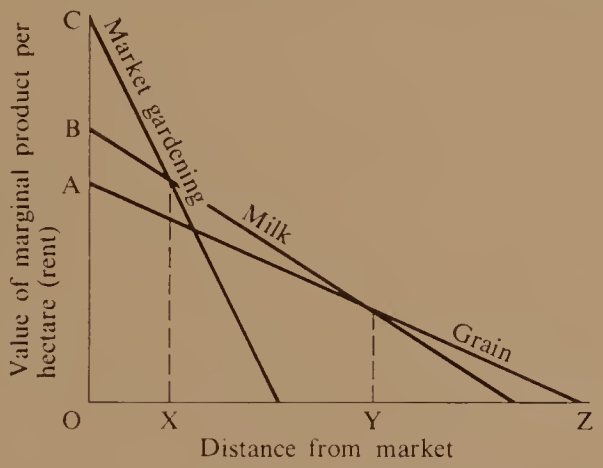


FIG. 5.9. The influence of location on land-use. If the transfer cost per kilometre and per £1 worth of market garden produce is higher than that for milk, which in turn is higher than that for grain, the decline in rent with increasing distance from the market will be most rapid for market garden produce and least rapid for grain as shown. The result is that market gardening yields the highest rent on land near the market (up to OX kilometres away), milk yields the highest rent over intermediate distances between OX and OY kilometres away) and grain yields the highest rent on the furthest land (from OY to OZ kilometres away).

production the value of the marginal product per hectare will rise. At the same time land which was previously sub-marginal for grain production will now be worth cultivating. This is shown in Fig. 5.8(b) by assuming a shift in the rent–distance relationship from the solid line, RM to the broken line R′M′. This shows a rise of RR′ in the value of the marginal product per hectare of all grain land and an extension in the area cultivated out to distance OM′ from the market centre.

Transfer costs also influence the comparative advantages of different areas in the choice of what to produce since some products can be carried more cheaply than others in relation to their market prices. The transport

cost per kilometre is likely to vary with the weight, bulk, perishability, and fragility of the goods carried so the price–distance gradient is likely to differ for different products. In view of the relative bulk or weight per £ of market value and the relative perishability of market garden produce, fresh milk, and grain, we might expect the product–price gradient to be steepest for the first and most gradual for the last of these three kinds of commodity. As a result the slopes of the rent–distance relationships for the corresponding alternative land-uses are also likely to differ as shown in Fig. 5.9. Again the relationships are assumed to be linear for simplicity and they are drawn on the assumption that the optimal intensity of land-use is achieved for each product at each location.

From this figure it is evident that differences in comparative advantage may arise as a result of differences in location. Thus land out to OX kilometres from the market centre has a comparative advantage, and earns a higher-valued marginal product, in market gardening; land from OX to OY kilometres away from the market centre has a comparative advantage in milk production; and land from OY to OZ kilometres away from the centre has a comparative advantage in grain production. Beyond OZ kilometres land is sub-marginal for all these uses.

These principles of the location of agricultural production are generally attributed to von Thünen, who published his theories in 1826.[5] He conceived the idea of a state which has no trade connections with any other nation, all produce being sold in a central city surrounded by a flat plain of uniform fertility. No lines of improved communication cross the plain and all goods are carried by a common means of transport. On the basis of these assumptions and the rent–distance relationships given in Fig. 5.9, a map of the area would show a series of concentric land-use rings—the so-called von Thünen rings—around the city, as shown in Fig. 5.10.

Obviously this is a purely theoretical, idealized model. Such land-use rings are unlikely to be observed in the real world for various reasons. First, many farmers may not choose the pattern of land-use which maximizes the value of the marginal product per hectare, especially if they do not actually pay an economic rent. For various reasons it may be desirable to diversify into several different enterprises on each farm. Secondly, the existence of variation in soil fertility will to some extent obscure the effect of location on land-use. Thirdly, the spheres of influence of adjacent market centres create supply areas of irregular shape. Fourthly, transport costs will not be the same in all directions but

[5] See Hall, P. (ed.) (1966). *Von Thünen's isolated state; an English edition of der Isolierte Staat.* Pergamon Press, New York.

will vary according to topography, the proximity of national frontiers or sea coasts, and the existence of canals, roads, and railways. Zones of supply are likely to extend outwards from major market centres along the main transport routes.

The construction of a new road or railway leads to a reduction in transport costs for all farms in the vicinity. As a result the supply zone for agricultural products is extended outwards, but at the same time, because of the saving in transport costs, all farmers in the area have an incentive to intensify production. Furthermore, as communications and

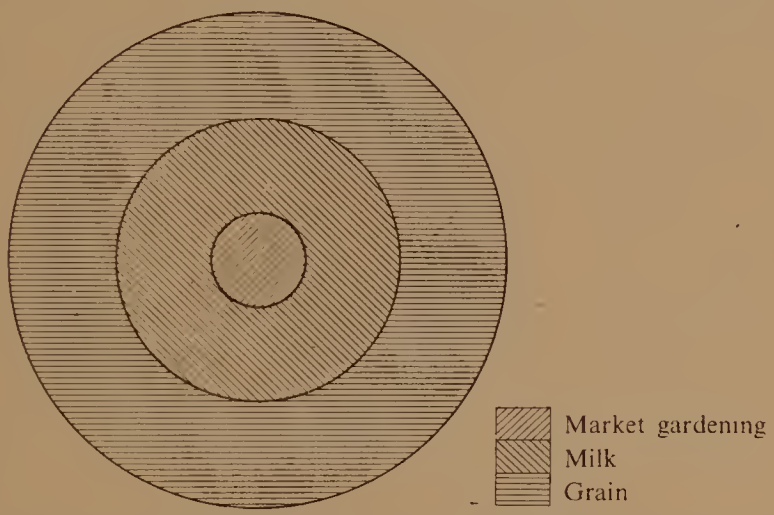


FIG. 5.10. Von Thünen rings. Given the relationships illustrated in Fig. 5.9 and assuming identical methods of transport, and hence identical transfer costs, in every direction from a central city market, we may expect the land-use zones to form concentric rings around the city. The inner ring (of radius OX kilometres) would be used for market gardening, the next ring (of outer radius OY kilometres) in milk production and the outer ring (of outer radius OZ kilometres) in grain production. In practice the number of transport routes is limited and the zones are much less regular, nevertheless they may often be discerned.

transport systems are improved, there is greater opportunity for specialization of production in different regions, each tending to concentrate on those crops and livestock most suited to the physical environment. Thus the better the transport system, the more closely may the pattern of land-use correspond to that best suited to local variations in climate, soil, and topography.

At a different level of analysis we may consider the main farm buildings as the centre of operations for the individual farm. In this situation the cost per hour of work by men and machines is likely to be greater on the more distant fields on account of the longer time spent travelling to and

from work. As the hourly cost rises so there is an incentive to work less hours per hectare of land. In other words there is an incentive to farm the more distant fields less intensively than those near the farm buildings.

Distance from the farm buildings may also influence the uses to which land is put. This is particularly obvious in the case of dairy farming where the twice- or even thrice-daily movement of milking cows from field to buildings and back again ensures that only those fields in the vicinity of the buildings are suitable for grazing by the milking herd.

It is conceivable, particularly if the greater difficulty of managing and supervising work on more distant fields is taken into account along with the longer time spent travelling to and from work, that there is a limit to the area of land that can be managed as a single unit under a given farming system. In this way the economics of farm scale may be related to location theories.

Land tenure

Land-tenure institutions are the laws or customs relating to the control and use of land. These institutions both influence and are influenced by the patterns of land-use and the rural population density. At the same time they also reflect the social and political characteristics of the nation. We are concerned here with the economic effects of different forms of land tenure.

There is such a wide diversity of land-tenure institutions in the world that classification is difficult.[6] However, if we ignore questions of whether farms are operated by single individuals, groups of individuals, or the state, and whether land ownership is vested in single individuals, groups of individuals, or the state, we may distinguish just five ways in which rights to use land are held. These are:

(1) *leasehold (fixed rent) tenancy*—under which a contractually agreed, fixed annual rent is paid for the use of the land by the tenant to the owner or landlord;[7]

(2) *share-tenancy*—which differs from a fixed rent contract in that the tenant agrees to pay a stipulated fraction of the harvest in return for use of the land instead of a fixed amount per hectare;

[6] For a classification which is claimed to be universally applicable see Denman, D. R. (1970). Proprietary patterns and land use. In *New possibilities and techniques for land use and related surveys* (ed. I. H. Cox). World Land Use Survey, Occasional Papers No. 9. Geographical Publications Ltd., Berkhamstead, Herts.

[7] Pledging, which is common in Africa, may be viewed as a special kind of leasehold tenancy in which rent is paid by the tenant in the form of an interest free loan.

(3) *freehold ownership*—under which the farmer as the owner pays no rent and is free to sell the land if he so wishes *or* (in most areas) to lease to others;

(4) *individual user right*—which differs from the above categories in that no regular annual payments are made for the use of the land and the operator is not free to sell it; and

(5) *communal (common) rights*—most often apply to grazing land, the use of which is shared by many farmers.

It would be wrong to assume that any one of these forms of land tenure is universally superior to another, partly because social and economic conditions vary in different areas of the world, and partly because there is more than one criterion by which land-tenure institutions may be judged. One desirable feature would seem to be that productive efficiency should be promoted. On the other hand there are many who believe that the system of land tenure should promote social justice. In fact, land-tenure reforms in many countries have been aimed at redistributing wealth and incomes rather than increasing agricultural output. In some circumstances both aims may be achieved but in other cases one is achieved at the expense of the other. The issues may become clearer as we consider each of the five forms of land tenure in more detail.

1. *Leasehold (fixed-rent) tenancy*

Under conditions of pure competition in the market for leasehold land where there are many potential users competing among themselves for its hire, productive efficiency will be encouraged in the following ways.

(a) The land will be allocated among potential users according to their ability to make productive use of it. Thus the man who can earn the highest marginal product on each hectare of land will be able to pay the highest contractual rent. In this way he will gain control of a larger area than his less well-qualified neighbours.

(b) Land will be put to its most profitable use, for if not the tenant will be unable to afford the rent payments. For instance, on land with a comparative advantage in dairying, competition will force rents up to a level which can be earned only by dairy farming. Other alternative uses, such as arable farming, will not provide an adequate return, on that land, to meet the costs of all other factors of production *and* the rent that can be earned by dairy farming.

(c) The land will be divided up into farms of the most favourable size, given the available range of managerial skills and other resources. Thus if, by amalgamating farms into larger-sized units, the value of

the marginal product for each hectare of land is increased, higher rents per hectare can be afforded on these larger-sized farms: there will be an incentive for the landlord to amalgamate holdings. Conversely if higher rents per hectare can be afforded on smaller farms, there will be an incentive for landlords to subdivide holdings before leasing to tenants. Whether this incentive will be effective or not will depend upon whether the rent increase is sufficient to compensate for the costs of subdivision. Thus the payment of competitive rents by tenants will provide a motive for land to be farmed at the economic optimum intensity. Competitive rents are likely to be higher in more densely populated areas, on relatively fertile land, and in areas nearer to main markets or transport routes.

Thus although the payment of rents may not affect the availability of different kinds of land, it does affect the way in which the available land is used. If for some reason rents are fixed at artificially low levels, at least a part of this organizational effect would be lost. To be sure, profits would still be maximized by growing those crops with the highest comparative advantage on each type of land. However, it would be possible for individuals unwilling or unable to make most productive use of the land to afford the rent, which is held below the competitive level. At the same time there would be less incentive for the landlord to divide up the land into units of optimum size. Indeed, there would be less direct guidance as to what was the optimum size.

In this context it should be noted that the landlord normally provides a certain amount of capital in the form of buildings, farm roads, fences, ditches, and so on, so that a part of the contractual rent is a payment for the use of this capital. Now, although the availability of a natural resource may not be affected by the price paid for its use, the availability of a particular kind of capital most certainly will be, although there may be a time lag of several years before the full effect is felt. This means that, if rents are held so low that landlords do not receive an adequate return on their capital, there will eventually be a reduction in the amount that they are able or willing to spend on 'landlords' capital'.

At the same time, the tenant provides the remainder of the farming capital and he too requires an adequate return to justify his expenditure. But it may be many years before he can reap the full return. Hence he needs the assurance of either security of tenure or else full compensation, in the event of his having to give up the tenancy, for any capital he provides. This kind of security can be provided adequately only by

contracts of tenancy which are enforcible by law. However, the provision of such safeguards for tenants may make it difficult to evict a tenant who is unable or unwilling to make best use of the land and thereby to earn a competitive rent. Thus the beneficial effects of providing security of tenure, in encouraging tenants to conserve and make permanent improvements to their farms, may be outweighed by the disadvantages of impeding the working of a competitive market in land.

From the point of view of social justice there are some who consider that rents are 'unearned incomes' which should therefore be subject to heavy taxation, and others believe the very existence of a landlord class to be iniquitous and favour the nationalization of all land. These attitudes oftem stem from the knowledge that rents may increase with rural population growth, the development of urban areas and transport routes, the establishment of new markets, and the provision of agricultural subsidies aimed at supporting tenants' incomes, without any extra effort or cost on the landlord's part.

In fact there is another, practical argument in favour of taxing rents, namely that in so far as the total area of available land is fixed, this will not be influenced by the land rent that the landlord receives. By contrast, the introduction of a tax on the price received by tractor manufacturers, for instance, is likely to cause them to reduce their rate of production so the number of tractors available per year would fall. Naturally, a differential tax on certain kinds of land-use but not on others would have a distorting effect. Thus if urban land rents but not agricultural land rents were taxed then it is likely that less land would be made available for urban development and more land would be used in agriculture than otherwise. However, a uniform rate of tax on all rents should, in theory, have no distorting effects on the over-all pattern of land-use.[8]

Those who do not favour special taxes on land rents might argue that rent is not 'unearned income'. Landlords provide a service: that of estate management. At the same time land is not the only factor whose supply is fixed independently of its price, as we have seen in earlier chapters. (It is often pointed out that stars of the stage, screen, and football pitch would probably be prepared to perform for far less than they are actually paid. Much of their income is a scarcity payment.) Secondly, as we have noted above, if landlords' rent revenues are reduced this may have an adverse effect on the provision of 'landlords' capital'. In fact the main problem in taxing rents is to distinguish the true rent, which in theory does not

[8] The most noted advocate for taxing land rents is Henry George (1839–97). See George, H. (1879). *Progress and poverty*. Appleton–Century–Crofts, New York.

affect the amount of land available from the other elements of contractual rent which do affect the provision of other factors of production. Finally, it is possible that taxation of land rents *would* affect the amount of land available for renting, since the landlord has the alternative of keeping the land for his own use. The land then has an opportunity cost; either the value of the marginal product the landlord could earn by farming each hectare himself, or the subjective value he puts on being free to use the land for his own leisure pursuits. If rent revenues from tenancies are reduced by taxation the landlord may well choose to keep more land for his private use.

A policy of land nationalization would of course eliminate the freehold ownership of farms but leasehold tenancies would still be possible with the State as landlord. In fact there are large areas of land in public ownership in most countries of the world, including Britain.[9]

2. *Share-tenancy*

Under a share-tenancy the rent payments vary from year to year according to the 'state of nature' and the resultant crop (or livestock) yields and prices. Thus the landlord takes a share of the risk, which is borne entirely by the tenant under a fixed-rent tenancy. Besides sharing the risk the landlord often provides a larger share of the capital, including working capital, and under share-tenancy may actually be involved in managerial decision making. Indeed, some share-tenancies are little more than a system of labour hire, the worker receiving a share of the crop instead of a fixed wage.

Many economists have argued that share-tenancy is inefficient in that it discourages tenants from expanding production up to the economic optimum level. This is illustrated in Fig. 5.11, which shows the marginal product per unit of labour and other variable inputs supplied by the tenant. The line marked marginal net revenue shows the marginal net return to the tenant, assuming that the landlord receives half the product as rent. The horizontal line WW′ represents the cost per unit of the variable inputs. Clearly, in terms of the analysis of Chapter 3, E_1 is the economic optimum level of variable inputs. We are of course ignoring the effects of risk and its sharing. However, the share-tenant maximizes his profit at E_2. Thus the land will be cultivated less intensively under share-tenancy.

In fact there is a flaw in this argument, since it considers only one

[9] See Gibbs, R. and Harrison, A. (1973). *Land ownership by public and semi-public bodies in Great Britain.* Miscellaneous Study No. 56, University of Reading Department of Agricultural Economics and Management.

party to the rental agreement, namely the tenant. The landlord is presumably also interested in the amount *he* will receive. Now, if variable inputs were increased to E_1, the whole of the *shaded* and *cross-hatched* areas would be a surplus over and above the tenant's costs. With a competitive market in fixed-rent tenancies the landlord would receive the whole of this as rent for his land. Alternatively if he farmed the land himself and hired in labour and other variable inputs he would earn this surplus. But if the share-tenant uses only OE_2 units of input the landlord's rent revenue is reduced to the cross-hatched area alone. In fact it is

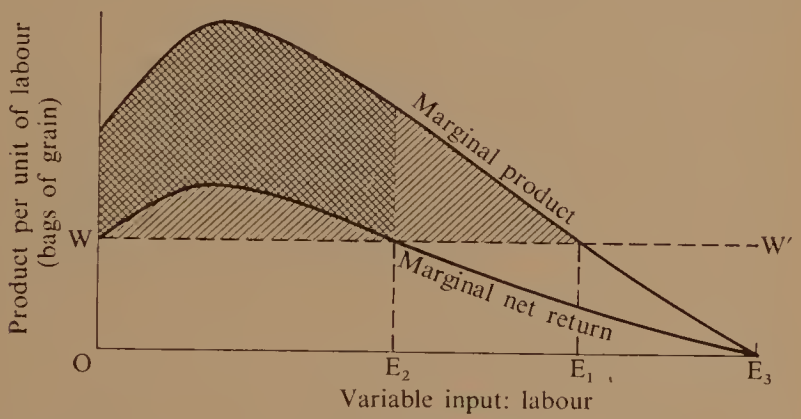


FIG. 5.11. Share tenancy and intensity of land-use. The marginal net return per unit of variable labour input is only half the marginal product, on the asumption that the landlord receives the other half as share-rent. It would appear that for a given wage rate (OW bags of grain per unit of labour) the economic optimum intensity of labour-use under share-tenancy is lower (at OE_2 units) than it would be under a fixed-rent tenancy (at OE_1 units). However, this would mean less rental income for the landlord, which he is unlikely to accept. Share-cropping will be accepted by the landlord only if for technical, legal, or economic reasons the tenant is forced to cultivate intensively.

unlikely that the landlord would consent to such an arrangement. Since he is sharing the risk with the tenant, if anything he is likely to want a larger rent revenue than he would receive under a fixed-rent tenancy.

Yet share-tenancies are common in many parts of the world, including Europe, the United States of America, India, China, and Africa. It is unlikely that this form of tenure would be so widespread if it always worked to the landlord's disadvantage. In fact, situations may be envisaged in which share-tenancy would not discourage tenants from expanding production up to the economic optimum at E_1. For instance if there is only one way of growing the crop which is shared, by a fixed-factor-proportions activity, the amount of labour and other variable inputs used

per acre would be constant regardless of the share of the crop taken by the landlord. The profit-maximizing level of the tenant's input would correspond with the over-all economic optimum.

Alternatively, the tenant may not be concerned with maximizing his profit over and above a fixed wage rate as shown in Fig. 5.11. For instance, if he provides the labour and other inputs himself and does not pay an explicit unit cost for these, he may be more interested in maximizing his total earnings. This would be achieved by increasing labour and other inputs until his marginal return per unit falls to zero. In this case the landlord might receive more than he would under a fixed-rent tenancy, as shown at E_3 in Fig. 5.11.

In other situations, however, it may be necessary for the share-tenancy agreement to be modified to encourage efficient production. For instance, the agreement may include a specification of the amount of labour and other inputs that the tenant will provide per hectare. Thus the tenant may be forced to expand production up to E_1 although he may prefer to limit it to E_2. Alternatively, the costs of the inputs may be shared between landlord and tenant in the same proportion as revenues are shared. The profit-maximizing level of inputs for the tenant would then coincide with the original economic optimum.

In fact the main conclusion to be drawn from this discussion is that share-tenancy is a form of management sharing or partnership rather than a method of hiring land. It allows for great flexibility in the proportionate contribution the landlord makes to the provision of capital and to managerial decisions and the proportionate share of the product and hence of the farm profit he receives.

3. *Freehold ownership*

An owner-occupier clearly has considerable security of tenure, which will encourage him to conserve and indeed to make long-term improvements to the land. However, it should be noted that even under this form of tenure the farmer will rarely be entirely free to use the land as he wishes. For instance there may be legal restrictions on the use of land for building purposes and the prohibition of 'bad farming', in the sense of allowing land to deteriorate and become infested with dangerous weeds or pests.

The latter restriction may be necessary because, although freehold ownership provides security of tenure, it does not always provide as strong an incentive to make the most productive use of the land as does a system of leasehold tenancy, which requires an annual rent payment to be made. Of course, for the freehold owner, land has an opportunity cost in terms of the rent it would earn if hired to someone else or in terms of the

return that would be earned if the land was sold and the proceeds used to hire other productive resources. However, as in the situation where rents are held below their competitive levels, the farmer is not *forced* to maximize his profits merely to be able to meet his contractual costs. The land owner can take advantage of his wealth to farm less intensively than he would need to if he had to pay a competitive rent.[10]

It is important to recognize that the sale value of a piece of land depends upon the annual rent that can be earned and not vice versa; i.e. the rent does not depend upon the sale value. Note that we are using the term 'rent' here to mean the value of the marginal product earned per hectare regardless of whether an actual payment is made. Now when anyone buys a piece of freehold land he acquires the source of a permanent stream of income in the form of annual rents. In theory, if all prices were likely to remain constant in the future, the present value of such an income stream would simply depend upon the prevailing market rate of interest on borrowed money. For instance, take land which is expected to earn an annual rent of £50 per hectare. Given a market rate of interest of 10 per cent it may then be argued that the market *value* will be £500 per hectare, since £50 is just 10 per cent of £500. If the price were more than £500, say £550, the annual rent would represent a rate of return of less than 10 per cent (£50 is approximately 9 per cent of £550). No one would be prepared to buy at this price since more could be earned by lending money out at 10 per cent interest. If, on the other hand, the price were less than £500, say £450, the rate of return would be more than 10 per cent (£50 is approximately 11 per cent of £450). It would then be profitable to borrow money at 10 per cent interest to buy land, and to earn a return of more than 10 per cent. Hence, so long as the price is less than £500 per hectare there will be an incentive to buy land but not to sell it. In technical terms the demand for land will exceed the supply (or there will be excess demand for land) at that price. As a result, in a competitive market, the price of land will be driven up to £500, when there is no longer any excess demand.

By the same reasoning, at a market rate of interest of 20 per cent, instead of 10 per cent, the value of this type of land would be only £250 per hectare, since £50 is 20 per cent (or 0·2 expressed as a decimal fraction) of £250. In fact the value is obtained by dividing the annual rent by the rate of interest, thus £50/0·2 = £250. Clearly the higher is the rate of interest, the lower is the value of land for a given annual rent.

[10] Note however that a freehold owner whose land is mortgaged, i.e. used as security for a loan, is in a position much closer to that of a leasehold tenant (see Chapter 7 on capital).

At any *given* rate of interest, it follows that the values of different plots of land will vary directly with the rents that can be earned. If land which earns £50 rent is valued at £500, then land which earns only £25 rent will be valued at £250. Hence, just as rents diminish with increasing distance from market centres, as shown in Figs 5.8(b) and 5.9, so too will land values. On this basis marginal and sub-marginal land which earns no rent will be valueless.

In practice this theoretical analysis is inadequate to explain the determination of land values. The first and most important reason is that rents can rarely be expected to remain constant. If rents should rise in the future so too will land values, and landowners will make capital gains.

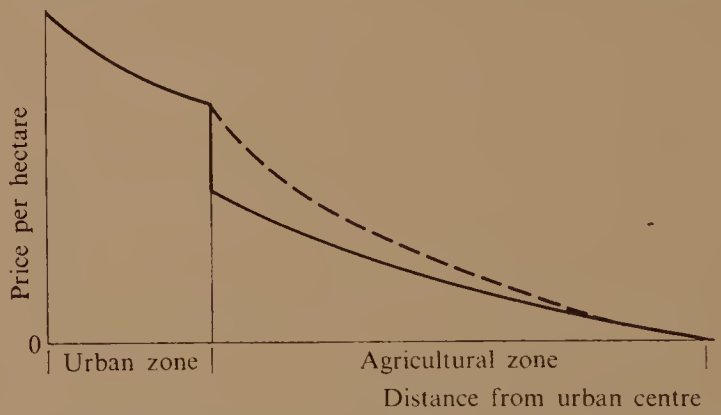


FIG. 5.12. Land-price gradient under land-use zoning. Since land prices are dependent upon land rents and since rents decline with increasing distance from urban centres, so too will land prices. In the vicinity of the urban centre, land is valued more highly for building than for agricultural use. Land-use zoning introduces a barrier which may cause a large discrepancy to develop between the price of building land and the price of agricultural land, as shown by the large step in the land-price gradient. Speculation that agricultural land may be redesignated for building may cause agricultural land prices to become artificially inflated, as shown by the broken line.

Conversely, if rents should fall landowners will make capital losses. Hence there is normally a speculative element in the prices paid for land. When rents are expected to rise the price will be higher than when they are expected to remain constant or fall. This means that initially the landowner's rate of return on his capital is lower than the market rate of interest, but he expects to recover these losses later.

The effect of speculative buying on land values may be greatly intensified by land-use legislation. For instance, if restrictions are imposed on the area of land which may be used for building purposes, this is likely to create an artificial shortage of building land and its price will

rise in relation to the price of agricultural land. As a result the minimum price per hectare of land designated for building may be considerably higher than the maximum price that could be justified by purely agricultural use. In the absence of speculative buying the price–distance gradient might appear as shown by the solid line in Fig. 5.12. However, if there is any prospect that at some future date building will be allowed on some of the agricultural land then its price will become inflated accordingly, for instance as shown by the broken line in Fig. 5.12.

The second reason why land value may differ from that dictated by current rent and interest rate is the relative security offered by land ownership. Generally speaking, in periods of inflation, rents are likely to rise along with other prices so that land maintains its value in relation to other resources, goods, and services, while of course money savings do not. At the same time rents do not fluctuate as widely as the annual returns on many kinds of business venture. Hence, individuals who are risk averse will be prepared to accept a lower average rate of return from owning land than they could expect to earn by lending money out at interest. In other words they will be prepared to pay a higher price per hectare than they would if they were indifferent to risk. On the assumption that most people tend to be risk averse, we may expect land values to be increased accordingly.

The third reason is that a certain amount of prestige is associated with land ownership, and it provides facilities for sporting and social activities. Furthermore, in some countries land is taxed less heavily than other forms of wealth. Thus for various reasons the value of farm land is commonly found to be higher than that justified by its current rental earnings. This means that those who buy farms with the help of loans may have severe difficulty in meeting the interest charges out of their farm income.

4. *Individual user rights*

This system of tenure, still found in India and Asia and widespread in Africa, gives the user considerable security. He cannot normally be dispossessed of his cultivated land and may pass on the rights of cultivation to his heirs. However, since the land is neither rented nor sold, there is no competitive market for its use. This does not matter where there is surplus unused land and cultivation is extended to the point where the marginal product per hectare is zero. Indeed, in such circumstances, customary user rights are probably the most appropriate forms of land tenure; there is no justification for rent to be paid or for land to have a purchase price.

However, once all the land of a particular kind is in use, every hectare will have a positive marginal product. Any use to which the land is put will then have an opportunity cost. If this opportunity cost is not reflected in an annual rent payment, or a corresponding purchase price, there will be less direct incentive to use land in the most profitable manner. Furthermore, in the absence of a market for land some alternative means will have to be found for allocating land between those competing for its use.

One approach that might seem attractive is to share the land equally among all potential users. Indeed, this approach is sometimes used on settlement schemes even though the land is purchased or rented from the Settlement Authority. However, such an approach will be relatively ineffective in promoting equality of incomes and must lead to inefficient use of available land. It will be ineffective in promoting equality of incomes because farmers vary both in the number of dependents they are supporting and in non-land resources, including their own managerial skills, at their disposal. Thus a farmer with a large family to support will require more land than a single man in order to achieve a given income level per person. At the same time a farmer with more resources and ability than his neighbour will undoubtedly earn a higher income even though he may be restricted to an equal area of land. Indeed he may choose to use his resources in non-agricultural, non-land-using forms of production.

Sharing land equally between users must lead to inefficiency in that more agricultural produce would be obtained by allowing the more able and resourceful farmers to acquire more land at the expense of the less able. It might nevertheless be considered desirable to reduce income inequalities even at the expense of some loss of total agricultural production. But it is more efficient, in terms of there being a smaller distorting effect on total production, to redistribute final incomes through taxes and grants rather than by placing a fixed limit on the amount of a particular resource, in this case land, which an individual may acquire.

5. *Communal (common) rights*

This form of tenure provides virtually no incentive to the individual user to make any improvements or even to conserve the existing productivity of the land. Although the costs are borne by the individual the benefits are shared by all the users. Hence in order to justify any improvements a change must be made either to communal farming, in which case both costs and benefits are shared by the community as a group, or to individual tenure so that neither costs *nor* benefits are shared.

Summary

1. The total quantity of land, and other natural resources, is fixed. It follows therefore that intensity of land-use will be higher (a) the greater the density of farming population, (b) the more fertile the soil, and (c) the nearer it is located to a market centre.

2. Generally speaking the marginal product or 'rent' per hectare will rise in relation to product price with increasing intensity of land-use. However, land-saving innovations may counteract this tendency.

3. Land which is only just worth using at zero marginal product is known as 'marginal land'. Production may be increased both by intensifying and by extending the margin.

4. It is profitable to put land to the use for which it has the greatest comparative advantage.

5. Leasehold tenure and the payment of competitive rents provides a direct incentive to use land as productively as possible. However the total supply of land is not affected by rent levels so in theory a tax on rent has no distorting effect.

6. Share-tenancy is in effect a form of partnership and risk sharing.

7. The price of land is dependent upon the marginal product per hectare.

Suggestions for further reading

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6. Labour

The measurement of work

Human labour has two separate functions: the first as a source of motive power and the second in regulating processes, as is the case when a man drives a tractor. In this latter capacity a worker may be referred to as a 'process control operator'. Human effort is also involved in farm planning and management but this function will be discussed separately in a later chapter. In practice, the farmer or manager may well do some physical work, in which case the separation of the two activities is difficult and somewhat arbitrary. Furthermore a certain amount of skill and judgement is exercised by every worker, both in manual work and in process control operation. Nevertheless we may draw a conceptual distinction between management, which has the function of planning the productive activity, and labour, which has the function of carrying out the plans, even though it may be impossible to identify these functions with specific individuals.

The theoretical economic optimum (maximum-profit) rate of labour-use was defined in Chapter 3. It occurs where the monetary value of the marginal product per unit of labour is equal to the wage rate or price per unit of labour. This conclusion depends upon the assumptions that labour is a homogeneous resource, all units are identical, that inputs of labour may be varied continuously without limit, that the prices per unit of labour and per unit of product are constant, and that certain other factors are fixed so that the law of diminishing marginal returns applies.

In practice it will be difficult to apply this theoretical analysis to planning farm labour-use. Difficulties arise because labour is clearly not homogeneous nor can inputs be varied continuously. But first there is a problem in defining a unit in which to measure labour inputs. The total amount of work done depends upon the number of persons employed, the number of hours each person works, and the amount of work done per hour. This last is particularly difficult to assess. For physical work it might be measured in terms of energy output but this would make no allowance for variations in skills and would clearly be meaningless in relation to process control operation. Alternatively, for a specific,

uniform task the work rate may be measured in terms of the output achieved. For instance land clearing and cultivation might be measured in terms of the area covered; crop processing tasks might be measured in terms of the quantity of produce handled. In such circumstances wages may be directly related to the amount of work done, by paying piece-work rates.

However, many tasks cannot be measured in this way and clearly the approach is unsatisfactory when each worker will carry out many different tasks from day to day or even from hour to hour as is the case in agriculture. Thus for most purposes labour inputs are measured in terms of man-hours (or man-days or man-years) and variations in work output per hour are ignored. Particular difficulties arise in areas where wives and children, who may lack the strengths and skills of the men, and who therefore achieve substantially less work per hour, are employed. Clearly in these areas labour is not homogeneous, and male, female, and child labour should perhaps be treated as three separate resources. On the other hand, these different sorts of labour may be perfect substitutes for certain tasks in which case it would be appropriate to treat them all as one resource. Standardized conversion factors are sometimes used to convert women and child-hours into standard man-hours, but the procedure is essentially arbitrary.

When we measure the labour input in man-hours we implicitly assume that the work of one man for 100 hours is equivalent to the work of 100 men for one hour or of 5 men for 20 hours: 100 man-hours in each case. This assumption is not satisfactory for all purposes. For instance in planning some farm tasks the gang size is an important consideration: for example a three-man gang may be needed for cereal harvesting. In such cases it may be more convenient to measure labour inputs in 'gang-work days'. At the same time it may be difficult to hire labour by the hour as and when required. Most labour is employed on a full-time regular basis, by the year, although there may be some men or women who are willing to take on temporary, casual work. (Agricultural contractors provide casual labour as well as machines on hire.) The distinction is important from a management point of view since casual labour is a continuously variable input while regular labour is not. Thus it is virtually impossible to adjust the rate of employment of regular labour so that the money value of the marginal product is *precisely* equal to the cost per unit. The economic optimum rate of employment is attained when the money value of the marginal product of each of the regular workers employed is greater than his wage cost but the extra revenue to be gained from employing one more would be less than the additional cost.

Farmers with motives other than profit maximization are still faced with the problem that regular labour can only be employed in whole units.

Characteristically a high proportion of the agricultural labour force is made up of family members; the farmers themselves, their wives, sons, and daughters. Most family members are like regular workers in that their services cannot be hired just as and when required; they are indivisible inputs. However, some may have alternative occupations and sources of income but can find time to help out on the farm at busy periods. These family members are a form of casual labour.

Generally speaking, the scope for employing casual labour will be limited. Given the choice, most workers would prefer a regular, full-time job rather than temporary casual work. Some might be prepared to accept temporary employment but only at a substantially higher hourly wage rate than they would earn in regular employment. At the same time temporary workers cannot be expected to develop skills and interest in the farm business in the way regular staff do. For most purposes it is helpful to treat casual labour as a separate and distinct resource from regular labour.

The seasonality of farm work

Farm work does not occur evenly month by month throughout the year; busy periods or 'work peaks' alternate with slack periods or 'work troughs'. Such peaks and troughs are illustrated by the tractor-work profile in Fig. 6.1, which is derived from records of the tractor-hours actually worked in each week of 1958 on a 75-hectare cereal farm in Berkshire, England.[1] No self-propelled machines were used, so this is also a profile of the field work done by the tractor drivers. On this farm, work peaks occurred in March during seed time and August–September at harvest.

Work peaks occur because certain critical tasks such as planting and harvesting are closely related to the seasons and must be completed within a limited period of time. Delays generally cause loss of yield, so the man-hours needed to finish the task are concentrated in the peak period. Other tasks, in particular maintenance and repair work, allow greater flexibility of timing and are therefore generally carried out in the slack periods. However the seasonal variation in the work load is rarely removed entirely. The organizational problems associated with the seasonality of farm work need not arise if casual labour is available at

[1] From Upton, M. (1963). *The use and management of farm motive power*. M.Sc. Thesis, University of Reading; also see: Upton, M. (1965). Labour machinery and profit on the farm. *Span* **8**, 47.

peak periods or if some other productive activity, cottage industry for example, can be carried on during the agricultural slack periods. However the scope for this kind of adjustment is generally limited.

For the family work force entirely dependent on agriculture, and for regular, hired workers, the supply of effort is relatively constant throughout the year. It is therefore probable that the farmer will either have less labour than he wants on the farm at work peaks, or more than he wants at slack times, or something of both. Some seasonal unemployment or under-employment is practically inevitable in agriculture. For instance, it would appear from Fig. 6.1 that two tractor drivers were required to

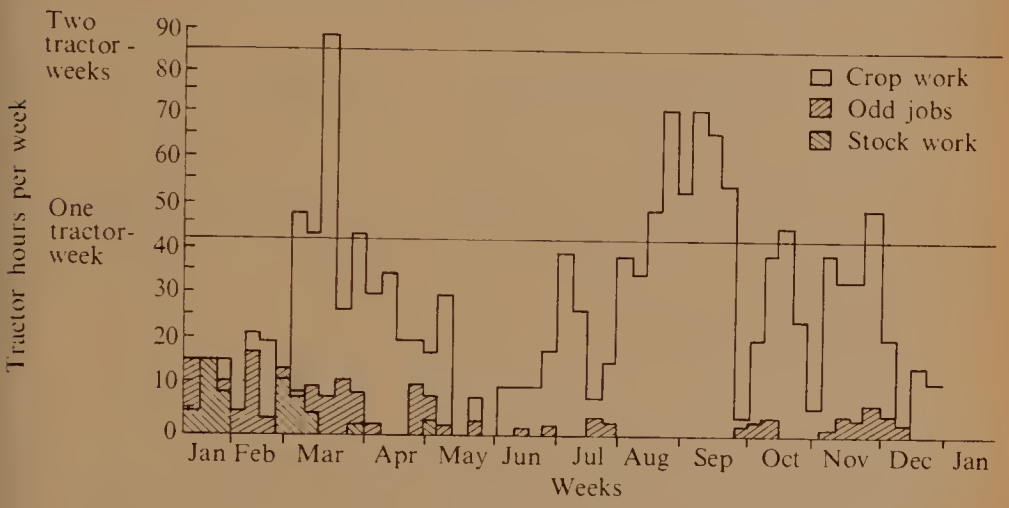


FIG. 6.1. Seasonal variation in field work on a 75-hectare arable farm. Because of the seasonal cycles of plant growth, agricultural field work inevitably occurs in alternating peaks and troughs. In particular peaks occur at planting (March and April) and harvest (August and September). Tractors, and probably men, are under-employed for much of the year.

cope with the peaks in field work on the farm in question. Over the year the field work amounted to a total of only 1528 hours, or about 30 hours per week, which if spread evenly over the year would be scarcely sufficient to keep one tractor and driver occupied full-time. Although tractor-drivers were presumably not idle when present on the farm but not driving their tractors, it is nonetheless likely that their efforts were less productive at such times than at work peaks.

In effect the marginal product and hence the opportunity cost per hour of labour varies from season to season. At work peaks an extra man-hour of labour input would yield a considerable increase in total product, either because of more timely completion of the work and hence higher yields, or because a larger area of crops could be cultivated. At slack

periods the marginal product of an extra hour of labour may be zero. There is no single, meaningful value for the marginal product or opportunity cost per man-hour of labour which applies throughout the year.

Although this seasonal pattern of work peaks and troughs is found on most farms, there is some variation between different crop and livestock production processes in the timing of these peaks and troughs. Thus the problem of allocating labour between alternative productive activities in the most profitable way is, in reality, far more complex than we have assumed. It cannot be achieved simply by allocating labour so that the value of the marginal product man-hour is equal in all uses, since labour

TABLE 6.1

Labour requirements per unit of output

	Activity 1: grain production; input requirement per bag	Activity 2: root production; input requirement per tonne
June labour (units)	0·025	0·01
December labour (units)	0	0·03

in the second week of June is not the same resource as labour in the first week of December or labour at any other time of the year.

The problem may be simplified by assuming fixed-factor proportions, as we did in Chapter 2. This implies that the seasonal distribution of man-hour requirements per hectare of crops and per head of livestock is constant regardless of the scale of each productive activity. On this basis we may compare the profiles of labour *requirements* for different crop and livestock enterprises. Now, there may be two or more enterprises which do not compete for labour at all because they are produced at different times of the year, in which case the enterprises are supplementary to each other. This would be the case, for instance, where there are two cropping seasons. Early-season crops would not compete for labour with late-season crops. Alternatively, enterprises may compete for labour in some months but not in others. This is illustrated by the hypothetical data of Table 2.6 (p. 55), some of which are reproduced here in Table 6.1. According to these figures grain and roots compete for labour in June but not in December. However, even when enterprises compete for labour in every month or indeed every week of the year, the rates at which they compete, that is to say the rates of product transformation, are likely to vary over the year.

In the circumstances, the labour profile for a combination of activities is likely to be flatter than that for a single activity; the seasonal peaks and troughs are likely to be smaller. At the same time the productivity of regular labour, measured by the value of the average product per man-year, will generally be higher. Hence it is useful in farm planning to estimate the profile of labour requirements for each proposed combination of enterprises. Such a profile will show whether a particular plan is likely to be feasible with a given labour force and will draw attention to the most probable incidence of work peaks.[2]

In any such analysis it must be remembered that weather conditions affect the number of hours available per man for field work in any particular month. So too will machine breakdowns and staff illness. Clearly some allowance must be made for such contingencies in estimating the man-hours available. Since weather conditions, machine breakdowns, and illness are somewhat unpredictable, this is a source of risk in farm planning. The farmer who is indifferent to risk will estimate his labour availability at its 'expected' value. He will then have surplus labour in favourable seasons and insufficient labour in unfavourable seasons but in an average year he will have just enough, assuming of course that his estimates are accurate. A farmer who is risk averse will assume fewer hours available per man-month, thus ensuring either a smaller total value product for a given labour force than is planned by his risk-indifferent neighbour *or* a larger labour force for a given total value product but less likelihood of having insufficient labour. Subjective judgement is necessarily involved in assessing labour availability for those who are not indifferent to risk.

The estimation of profiles of labour requirements does not lead in any systematic way to an optimal or most profitable allocation of labour among enterprises. For this purpose linear programming is a suitable tool. Its usefulness in this respect is illustrated in Chapter 3 (see Tables 3.10, 3.11, and 3.12; pp. 114, 115, and 116 and Fig. 3.10; p. 112). The example problem given there includes two alternative productive activities, grain and roots, and three labour constraints, one on the supply in June, one on the supply in December, and the other on the total supply. A problem involving many more alternative productive activities and 12 monthly labour constraints could be set out in a similar manner and solved by linear programming (though not by means of a two-dimensional graph as used in Fig. 3.10). Indeed, 52-weekly labour

[2] The 'gang-work day chart' is a more sophisticated version of the labour profile. See for instance Ministry of Agriculture, Fisheries and Food (1969). *The farm as a business: aids to management, labour and machinery* (2nd edn). H.M.S.O., London.

constraints could be incorporated but this might imply an unrealistic rigidity in the application of labour inputs, since in practice there is some flexibility as to timing.

This exercise merely gives the optimal allocation of a given labour force; it does not determine how many men should be employed. However, linear programming may give a solution to this problem. If the computer programme allows certain variables to be restricted to integer values, i.e. if an integer programming option is available, solution is straightforward. Again for illustration a very simple, hypothetical example is given in Table 6.2. With a suitable computer programme a

TABLE 6.2

Linear (integer) programming the optimal regular labour force

	Constraint level	Grain production	Root production	Hire casual labour in September	Hire regular† labour
Gross margin or cost per unit (£)		10	35	0.50	250
Land (hectares)	70	1	1	0	0
March labour (man-hours)	0	8	8	0	−100
June labour (man-hours)	0	10	4	0	−100
September labour (man-hours)	0	8	10	−1	−100
December labour (man-hours)	0	0	16	0	−100

† Integer variable

solution may be obtained directly. Even without an integer option a solution could be obtained by repeatedly running the programme, first with the regular labour force constraints set at say 4 men, then 5, then 6, and so on, until the most profitable of the optimum solutions is found. This approach is known as 'parametrization'.

The particular example given in Table 6.2 is so simple it may be solved by intuitive methods or using a two-dimensional graph as in Fig. 6.2. The reader may check that September labour *would* be an effective constraint with either 4, 5, or 6 regular employees in the absence of casual labour but that by hiring casual labour in September the constraint becomes ineffective. For 4 men the optimum solution is to grow 25 hectares each of grain and roots yielding a total gross margin of £1125, March and December labour being the effective constraints. Subtracting the wage costs of £1000 for four regular staff and £25 for 50 hours of

casual labour in September leaves £100 profit. For five men all costs and revenues are merely raised by one quarter. However when six men are employed the land constraint becomes effective and the net gain (gross margin minus *casual* labour costs) of employing the sixth man is only £245

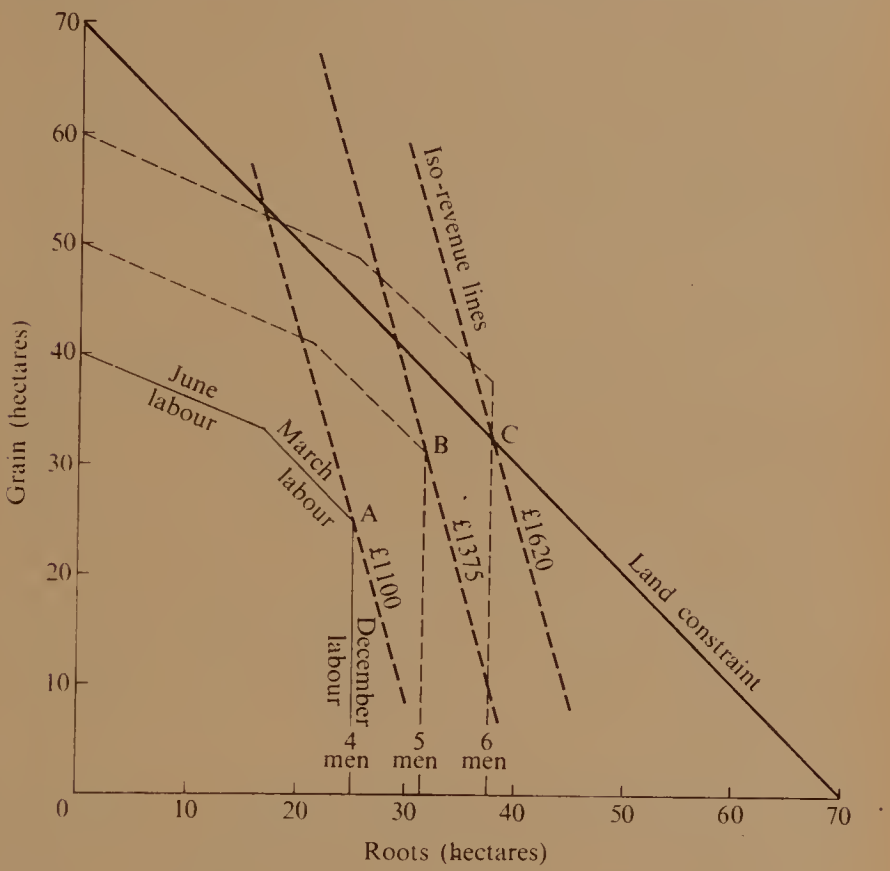


FIG. 6.2. The optimum regular force. For a regular labour force of 4 or 5 men the feasible area is bounded by the labour constraints in June, March, and December. At the economic optimum combinations of roots and grain (points A and B) March and December labour are the effective constraints. However, when 6 men are employed the land constraint becomes effective (point C) and labour is left in disposal in March as well as June. The difference in revenue between points B and C is less than the cost of employing the sixth man, so the optimum regular labour force is 5 men.

while the wage cost is £250. Thus it does not pay to employ this worker and the optimal labour force is five men.

This simple example serves to illustrate the importance and the complexity of determining the most profitable allocation of each alternative size of regular labour force before it is possible to determine the optimum number of men to employ.

Increasing labour productivity

The analysis so far has depended upon the assumptions of fixed labour requirements per unit of each activity and a fixed number of hours available per man per month. Now in reality there may be scope for varying on the one hand the hours each man works, and on the other hand the man-hours used per hectare of each crop or per head of each kind of livestock. We will return to the question of what determines the number of hours worked per day in the next section. Here we are concerned with the possibility of varying labour inputs per unit of output.

According to the theory set out in Chapter 1 it should be possible to vary the amount of labour used with given quantities of land, machinery, and other fixed factors in any particular productive activity. If this is so, then work peaks may be reduced and labour productivity may be increased without changing the combination of crops and livestock grown. Naturally, labour-saving methods involve a cost which must be compared with the gain. For instance, we may refer to the example given in Table 6.2, and now assume that the March labour requirement for grain may be reduced to 6 man-hours per hectare at a cost of £1 per hectare. This means that more grain can be grown with a given labour force though the gross margin per hectare is reduced from £10 to £9 per hectare. With four regular workers it turns out that $33\frac{1}{3}$ hectares of grain may now be grown in place of the 25 hectares previously estimated. The profit turns out to be £$116\frac{2}{3}$ as compared with £100 without the labour-saving technique.

Labour may be saved at peak periods either by (a) omitting certain operations, (b) extending the time period over which operations are spread, or (c) increasing the rate of working. The possibilities of omitting certain operations are clearly limited but it is sometimes possible, for instance, to omit some cultivations without excessive loss of yield. Then again the use of herbicides may eliminate the need for weeding operations. Labour may be saved by burning rather than harvesting cereal straw. Labour peaks may be reduced by spreading the critical operations over a longer period. Thus an early start to land cultivations may enable a larger area to be cultivated and sown with a given labour force.

There are several ways of increasing the rate of working at peak periods, in particular: (1) mechanization, (2) improvement of working methods, and (3) incentive payments. Generally speaking, mechanization represents the substitution of capital for labour. Although crop yields per hectare may be increased by using machines, on the whole they are labour-saving innovations. Mechanization is not just a case of using a machine in place of hand work. A larger and more complex machine will

produce a higher work output per man-hour of labour. Hence there is usually a range of opportunities for substitution based on machines of different sizes. The size of machine or level of mechanization that can be justified financially must depend upon the relative costs of men and machines.

'Work study' is the formal name given to the search for improved working methods ('method study') and to the timing of rates of work ('work measurement').[3] Method study may be used to simplify the routine used by a single individual worker or to reduce waste time by reorganizing work gangs. Although method study is likely to be more effective in saving labour if carried out systematically by a skilled work-study specialist this will obviously cost more than if the farmer uses his own judgement. Apart from the cost of making the study, new methods may involve new equipment, or modifications of the farm layout and training of staff. All these cost items may be regarded as forms of capital, the last sometimes being referred to as 'human capital'. Thus the improvement of working methods also involves the substitution of capital for labour.

Work measurement, as we have seen, is necessary if the worker's pay is to be related to his work output, so that he has a financial incentive to increase his productivity. Such 'payment by results' may take the form of either a simple piece-work rate whereby the wage paid is directly proportional to the work output, or some kind of premium bonus system under which the wage rate per unit of output varies with the amount produced. As an example of the latter a ploughman may be paid a basic time wage for ploughing anything up to say 3 hectares per day. If he ploughs more than 3 hectares, however, he is paid a bonus of so much per hectare on top of his basic wage. Such a method of payment may be more effective than an ordinary piece-work rate in encouraging a high rate of working, but it will have a distorting effect on the worker's choices. The payment of overtime rates has a similar effect, which will be discussed more fully in the next section.

Payments by results can be used only for tasks (1) which are easily measured, (2) for which the rate of output is directly under the worker's control and not dependent on outside factors like soil conditions, or the speed at which other men or machines are working, and (3) where the quality of work is unimportant. Obviously quantity may be achieved at the expense of quality, so where the latter is important it is better to avoid incentives to increase output. The main disadvantage of payment

[3] For further details see, for instance, Harvey, N. (1958). *Farm work study*. 'Farmer and Stockbreeder' publication.

by results is that it causes variation in incomes both from week to week over the year and between individuals. To some extent both forms of variation may cause discontent among workers.

In fact, since contented workers are likely to be more productive than discontented ones, the provision of good working conditions, the removal of drudgery, and the establishment of happy working relationships between employer and employee may be the most effective incentives for higher productivity.[4]

Income and hours worked

Each hour that a man works represents an hour less available for leisure. Presumably he derives a certain amount of satisfaction from his leisure, be it spent in sleeping, recreation, study, decorating his house and garden, or just idling. Thus work has a cost to him, namely the value he places on the leisure forgone. Although he may enjoy work for its own sake, his main reason for working is to produce goods and services or the cash necessary to buy them. For simplicity we will assume that all these goods and services can be evaluated in money terms and the total expressed as his income. Hence there is a trade-off between income, representing goods and services, and leisure.

For independent farmers and casual workers who may be free to choose the number of hours they work, the choice will depend in part on their personal evaluation of leisure in relation to income and in part on their rate of earnings per hour. To illustrate this let us consider the simplest possible case of a casual worker who earns a fixed hourly wage so that his marginal rate and his average rate of earnings are equal. The alternatives open to him are represented by the straight line OM in Fig. 6.3.[5] At point M he would work all his waking hours and earn a maximum daily income (at a wage of say 50p per hour this would be £9 for 18 hours work). At the origin he would do no work and therefore earn nothing. Other points on the line represent intermediate combinations of income and leisure. The horizontal line SS′ represents the minimum income necessary for subsistence. Now, if our worker enjoys his leisure and has limited aspirations he may choose to work the minimum necessary to survive (4 hours a day on our assumptions), assuming that he has no other source of income. The combination of income and leisure he will choose is represented by point A. Another worker, paid exactly the same hourly wage, who is more ambitious and energetic than the first, will choose

[4] See Upton, M. and Giles, A. K. (1973). It's more than a matter of £ p. *Big farm management*, p. 45–6 (October 1973).

[5] Readers may note the similarity with Fig. 2.1 (p. 37).

less leisure and a higher income. His choice may be represented by point B.

A rise in the wage rate may have two opposing effects. On the one hand a man is tempted to work more hours, since the marginal return for each extra hour is increased. This means that the opportunity cost of an hour

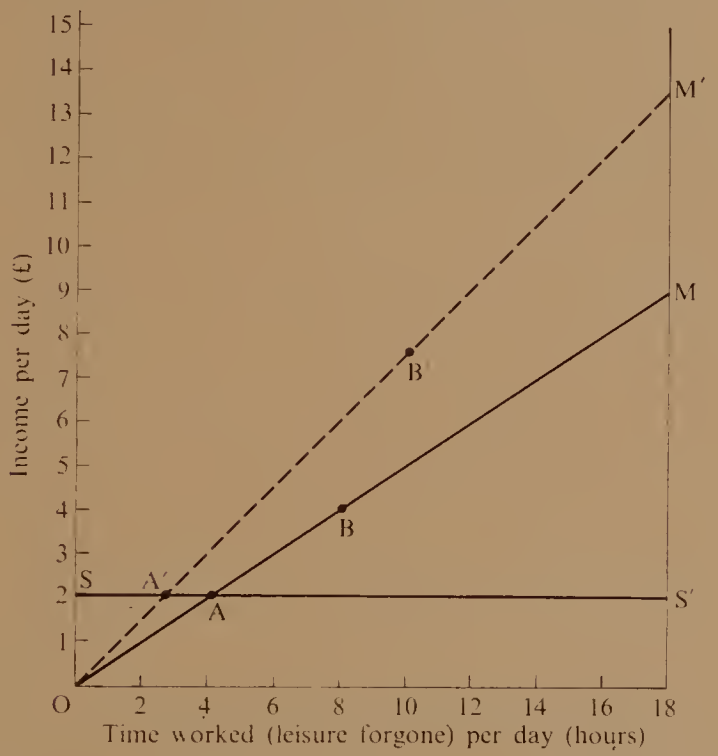


FIG. 6.3. The choice between income and leisure. It is assumed that men are free to choose the number of hours they will work each day up to a maximum of 18 (allowing a minimum of 6 hours for sleep), at any given, constant, hourly wage. The minimum income necessary for subsistence is £2 per day shown by line SS′. For a wage of 50p per hour, OM is the 'budget line' showing all feasible combinations of income and leisure open to the individual. One with a high preference for leisure may choose to work the minimum necessary to survive (point A) while another may choose more income and less leisure (point B). If the wage rises to a rate represented by the broken line OM′, there may be an 'income effect' (the individual uses the extra income to 'buy' more leisure e.g. he moves from A to A′) or a 'substitution effect' (the individual substitutes work for leisure since the latter is now more costly e.g. he moves from B to B′).

of leisure is greater than before, hence he is tempted to substitute work for leisure. On the other hand, operating against this 'substitution-effect' there is an 'income effect'. With a higher wage rate a man is better off and can afford to 'buy' more leisure. In Fig. 6.3, the opportunities

offered by a rise in the wage rate are shown by the broken line OM′. Our first worker takes advantage of this wage rise to take more leisure. He can earn his subsistence needs by working fewer hours as shown at point A′. His response is a pure income effect in that he consumes the whole of his income rise in the form of leisure. The other worker, however, prefers to substitute extra income for leisure. Indeed this substitution effect outweighs the income effect for him so he actually works more hours than before as shown at B′.

The over-all effect of the rate of earnings on the supply of effort is difficult to predict. A purely hypothetical supply curve is shown in Fig. 6.4.

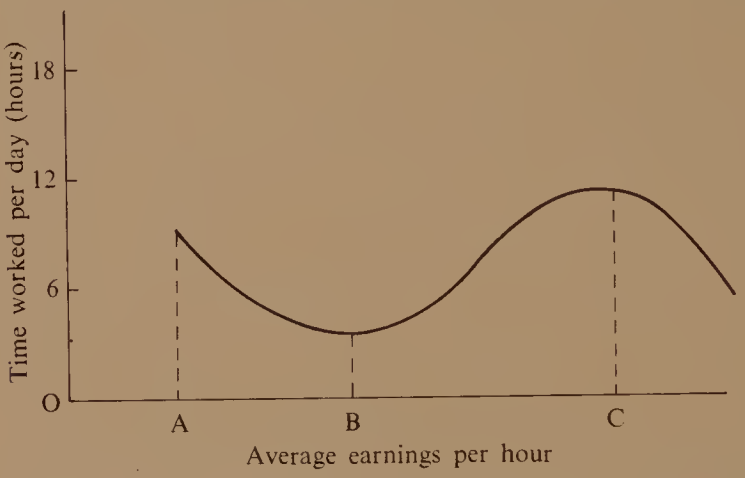


FIG. 6.4. Hypothetical supply curve of effort. At low levels of average return per hour (between OA and OB) an increase may lead to a reduction in the number of hours worked (the 'income effect' is prevalent and the supply curve is 'backward bending'). At intermediate levels of return per hour (between OB and OC) an increase may stimulate men to work longer hours (the 'substitution effect' prevails and the supply curve is 'normal'). With high hourly returns (more than OC) the curve may slope backwards again.

This curve is divided into three main stages above an hourly income of OA, which is assumed to be the minimum necessary to provide sufficient energy intake for a man to do any work. With earnings between OA and OB, the worker is just managing to subsist. His aspirations are limited, so once he has the basic necessities for life he prefers leisure to further effort. As his average hourly earnings rise he can achieve his subsistence income by working fewer hours. The income effect prevails and the supply curve is said to be 'backward bending'.

At a rate of earnings of OB, a man is working relatively few hours in order to survive. Further rises in earnings offer prospects of a higher standard of living and he is encouraged to work longer hours. At this

stage the substitution effect outweighs the income effect and the supply curve slopes forward. With hourly earnings of OC, a man is comparatively wealthy and working long hours. He may well prefer to take advantage of further increases in his income per hour by increasing his leisure time.

For farmers and their families, earnings per hour are likely to vary with the hours worked so the marginal return per hour of labour will differ from the average returns. The leisure/income opportunities line will look more like Fig. 1.1 (p. 10) or Fig. 2.10 (p. 53) than Fig. 6.3. However, this makes little difference to the analysis unless labour-saving or labour-using technological changes occur. A labour-saving innovation increases the average product (return) more than it increases marginal product. Hence total income will rise more than the opportunity cost of leisure, the income effect is likely to outweigh the substitution effect and less hours will be worked.[6] Conversely a labour-using change, which increases the marginal return to labour more than it increases total income, is likely to encourage more effort on the part of farmers and their families.[7]

When a hired worker is offered overtime rates of pay for hours worked outside his normal hours, the marginal return for each additional hour of work is raised and so too is the opportunity cost of leisure. Thus he has an incentive to substitute income for leisure through additional hours of work. On the other hand, his income does not rise unless he chooses to work overtime, so there can be no income effect. As a result, the offer of overtime rates will generally induce an employee to work more hours than he would do at the basic hourly rate of pay or indeed at the average hourly rate of pay he earns by working overtime. For example, if he works 8 hours at 50p per hour and 4 hours overtime at 75p per hour, his total daily income for 12 hours work is £7, an average of $58\frac{1}{3}$p per hour. Now, if the marginal return per hour of work were $58\frac{1}{3}$p rather than 75p, it is most unlikely that he would be prepared to forgo so much leisure. He would not work 12 hours per day.

This is illustrated in Fig. 6.5, where the straight line OBM represents the opportunities offered by the basic rate of pay of 50p per hour. The man chooses to work 8 hours and earn £4 per day as shown at point B. Now, he is offered overtime work at 75p per hour, represented by the steeper line

[6] This presumes that the farmer receives the whole of the average product of his labour and does not have to pay for the hire of other resources.

[7] Along similar lines it is argued that in theory a progressive income tax has a substitution effect and may discourage effort whereas a lump-sum tax has a pure income effect and encourages men to work harder.

BC. He chooses to work 4 hours longer to earn £3 more as shown at point C. At the average wage rate he earns at point C, shown by the broken line OC, he would undoubtedly prefer to work fewer hours, as at point D for instance.

The payment of an overtime rate, which differs from the basic wage rate, is a form of price discrimination which will have distorting effects on the employment of labour.[8] It causes employees to work longer hours,

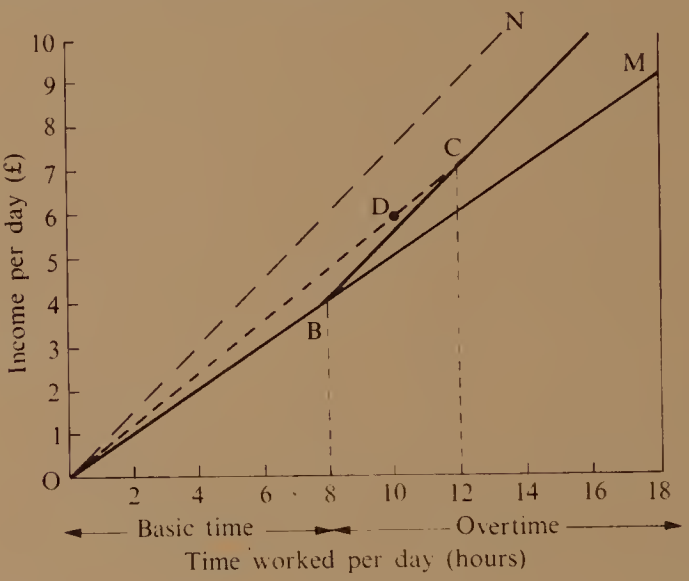


FIG. 6.5. The effect of overtime rates of pay. The alternatives provided by a constant hourly wage of 50p are represented by line OM as in Fig. 6.3. We assume an individual chooses to work 8 hours per day for £4 as shown by point B. Now if overtime work at 75p per hour is offered, represented by the line BC, the substitution effect is likely to outweigh the income effect and he will choose to work longer hours (4 more in this diagram). The substitution effect is as strong as if the basic wage had been raised to 75p as shown by broken line ON (the cost of an hour of leisure is now 75p) but the income effect is much smaller. At an equivalent, constant hourly wage as represented by the broken line OC, he would undoubtedly work fewer hours in total, say 10 (point D).

for a given average return per hour, than they would choose to do if paid a uniform hourly wage. In order to persuade men to work as many hours, at a uniform hourly wage, the employer would have to pay them considerably more in total; the average wage would have to be increased. Indeed it may be impossible to persuade men to work as many hours at a

[8] In fact the payment of overtime rates is possible only under conditions of monopsony, or buyer's monopoly. Under competitive market conditions the wage rate would necessarily be uniform for every hour worked.

uniform hourly wage rate. For instance, given the example illustrated in Fig. 6.5, it might be necessary to offer the man £1 an hour to induce him to work 12 hours, in the absence of special overtime rates. Alternatively, the man may not be prepared to work as long as 12 hours per day whatever the uniform wage offered, as shown in Fig. 6.4. The payment of overtime rates also distorts the employer's decisions since he is induced to employ *less* labour than he would at the same average wage if it were paid at a uniform rate, because the marginal cost is inflated.

In practice workers, even the self-employed, are rarely free to choose exactly how many hours they will work. For the organization of work gangs and other managerial considerations it is more convenient if the whole staff work similar hours. Furthermore, the risk of both employer and employee is reduced if hours worked and wage rates are fixed contractually in advance. Overtime work is offered only at work peaks. Consequently the hours of work are simply part of the contract the worker makes with his employer which, combined with the wage rate, payments in kind, and fringe benefits, determine the over-all satisfaction derived from his employment.

The influence of changes in incomes on the proportion of the population who seek work (generally known as the participation rate) is just as unpredictable as the effect on hours worked. On the one hand, as men become wealthier they can afford to keep their children at school longer, to relieve their wives of the necessity of taking paid employment, and to retire earlier. On the other hand higher rates of earnings may encourage more people to seek paid employment. However, a rise in the relative earnings in one industry will increase the supply of effort to that industry by attracting workers from other industries.

Population growth

In order to simplify our discussion of the relationship between population growth and agricultural output we will ignore the indivisibility of labour, the seasonality of farm work, and the scope for varying the supply of effort by each individual. The omission of these considerations should not affect the general conclusions. We are now concerned with the average and marginal product per man (per year), measured in units of food: tonnes of grain, for instance.[9]

It will be recalled from the previous chapter that the classical econo-

[9] Neither a balanced diet nor a 'balanced' farming system will include grain alone but the use of tonnes of grain rather than £s to evaluate output serves to remind us that we are here concerned with food production. We ignore the problems of aggregation implicit in either method of evaluation.

mists assumed the total area of cultivable land to be fixed so that in the absence of technological change, growth of the population engaged in agriculture brings diminishing marginal and average returns per person. Indeed it was thought that there is a natural tendency for population to outstrip food supplies, thus causing widespread poverty and perhaps famine. These dire predictions appear not to have been fulfilled in many of the high-income countries, but in some less developed areas population does indeed appear to be pressing on the means of subsistence.

It is widely assumed that, in such circumstances, family farming will provide more productive employment than any system involving hired

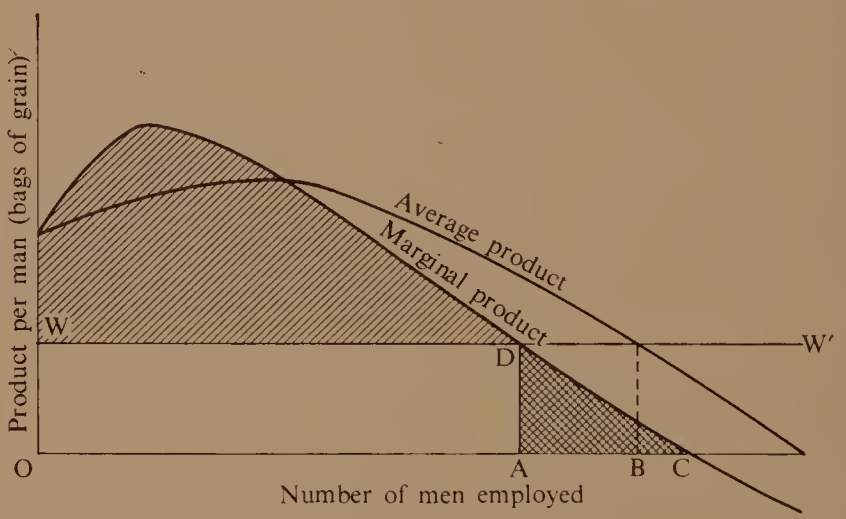


FIG. 6.6. Disguised unemployment. The basic subsistence requirement per person is represented by the horizontal line WW′. The surplus (the shaded area below the marginal product curve above this line) is maximized when OA men are employed. However, if this surplus can be redistributed equally among family members, then a total of OB men could be supported. Note also that total product is maximized (marginal product is zero) when OC men are employed.

labour. This is because hired labour will not be employed beyond the level at which the marginal product of the last man is equal to the wage rate, a rule which does not apply when no wages are paid. In the latter case the level of employment must be determined in some other way. One theory is based on the assumption that the family farmer's aim will be to maximize the total product, from which to support his family. He will then increase labour inputs to the level where the marginal product of the last unit falls to zero. However, this takes no account of the subsistence costs of family members or the subjective costs of leisure forgone. An alternative theory is based on the assumption that the total

product is shared equally between family members. This means that employment on the farm can be extended to the point where the *average* product per person has fallen to the minimum necessary for subsistence.

These theories of 'disguised unemployment' are illustrated in Fig. 6.6, which shows the average and marginal product curves for a typical farm. The horizontal line WW′ shows the wage rate for hired labour which is at subsistence level. The economic optimum rate of employment of hired labour is OA, where marginal product equals the wage rate. However, total food output is maximized at point C where the marginal product of labour is zero. The extra employment is the amount AC and the extra production is represented by the cross-hatched triangle ACD. According to this diagram, however, it is not possible to feed OC units of labour. The maximum number of persons that can be maintained on this farm is OB where the average product per person has fallen to subsistence level. In either case, at levels of employment of OB or OC, the marginal product is less than the subsistence wage; this is the main characteristic of disguised unemployment.

Disguised unemployment is unlikely to exist where there are alternative opportunities for wage employment, at or above the subsistence level. In that case labour has an opportunity cost. The farm family as a whole will be better off if employment on the farm is limited to the economic optimum level (OA in Fig. 6.6) and any additional family members seek work outside. Note too that if land is rented or any other inputs are hired it is impossible for each family member to receive the whole average product; part of it must go to meet these other costs. Hence the scope for the existence of disguised unemployment is limited.

In any case it represents a desperate state of affairs. If the average product per person is barely sufficient for survival then any further population growth must cause misery and famine. Technological change or additional capital might shift the response curve upwards, to improve incomes but so long as there is disguised unemployment there is no surplus from which to finance technological change or new capital. In Fig. 6.6 the shaded area above the line WW′ is the surplus above basic subsistence needs, when the economic optimum (OA) units of labour are employed. With disguised unemployment this surplus is dissipated in feeding additional family members.

This analysis is not particularly helpful since it is essentially static. Population (labour force) is the only factor which is assumed to vary. In all but the most desperate situations it is likely that capital accumulation and technological innovation will occur over time. Note that for present purposes the clearing or drainage of new areas of land may be treated as a

part of the process of capital accumulation. Note too that economies of increasing farm size may be obtained as capital is accumulated. Therefore the three factors (1) capital accumulation, (2) technological innovation, and (3) economies of increasing size, the effects of which may be extremely difficult to separate, will cause the production-response curve for labour to shift upward. Whether or not this causes average incomes to rise depends upon the relative rate at which the population is growing.

This is best illustrated with the hypothetical figures for a single farm given in Table 6.3. Fig. 6.7 is based on similar data although it shows

TABLE 6.3

Increase in labour force and corresponding increase in total product

Increase in labour force (per cent)	Increase in total product† (per cent)
0	0·50
0·5	0·88
1·0	1·25
1·5	1·63
2·0	2·00
2·5	2·38
3·0	2·75
3·5	3·13

† Based on the production function $Y = K^{0·25} L^{0·75}$ and assuming capital is growing at 2 per cent per year.

the effects of maintaining the same *linear* growth rates over 10 years, simply to make the drawing clearer. These figures show that the rate of capital accumulation and other changes are such as to increase the total product by 0·5 per cent, even if there is no change in the labour force. This means that the average product per man could also rise by 0·5 per cent. If, however, the labour force should also increase, the total product will be increased still further, but the change in average product will diminish. Thus we assume that a given increase in the percentage population growth not accompanied by any *extra* capital or technological changes will add a somewhat smaller amount to the percentage increase in total product. Under these assumptions there will be a certain percentage rate of increase in the labour force at which the average product per man will remain constant. It is 2 per cent per year in the present example. This is known as the 'critical expansion rate'.

Now, to shorten the discussion, let us assume that there are no alternative employment opportunities and that the farm labour force remains a fixed proportion of the whole farm family. If the family size (farm population) grows at exactly the same rate as the critical expansion rate then a kind of balance will be maintained between population and food supply, but it will be a highly unstable balance. Consider first the

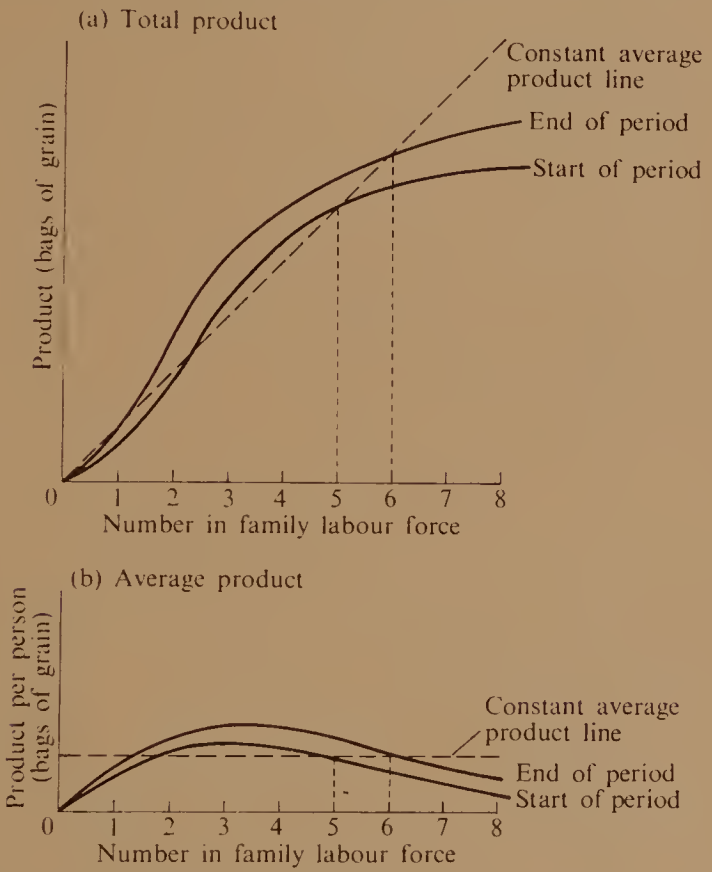


FIG. 6.7. Growth in population and output. Over a given period, of say 10 years, technological change and capital investment will cause the total, and average, production response curves for labour to shift upwards as shown. There are 5 units of labour at the start of the period so the broken line joins all points at which the average product per person is equal to that at the start of the period. Thus if, by the end of the period, the labour force grows to less than 6 units the average product per person will rise whereas if it grows to more than 6 units the average product per person will fall.

situation in which family size tends to grow faster than the critical expansion rate, in which case average product per person is falling. This may be referred to as a case I situation. It may be possible to feed more people, up to a point, by consuming more of the total product, thereby

leaving less for creating new capital. But the effect of this will be to reduce the critical expansion rate. The only possible result, if the case I situation persists, is that falling food consumption per head will cause the death rate to rise so that the population growth rate will fall. Thus the *actual* population growth must fall below the critical expansion rate. Eventually a static situation of disguised unemployment may result. Any small injection of capital from outside, or technological change which raises output, will simply enable more of the family to survive, without any significant improvement in living standards. Only if the capital or technological change is sufficient to raise the critical expansion rate above the potential rate of population growth will a self-sustaining improvement in income be possible. Hence this analysis is sometimes referred to as the theory of the 'critical minimum effort'.

In the case II situation, family size is growing at less than the critical expansion rate so that the average product per person is rising. Hence the surplus over the basic food needs is rising and the surplus may be used (a) to purchase more non-agricultural products and (b) to accumulate capital and thereby to raise the critical expansion rate. If the critical expansion rate rises, and the population growth rate remains constant, *per capita* incomes will rise yet more rapidly. Thus self-sustaining growth results.

This analysis may be applied to the whole national economy, assuming again for simplicity that there is no external trade or population migration. Indeed it may be applied to the whole world, our 'spaceship earth', in which case these assumptions are clearly valid.[10] Now, however, inequality of income distribution must also be taken into account. For instance, in a case I situation where *potential* growth of total population exceeds the critical expansion rate for the whole economy, it is perfectly feasible for average income per head of population to rise.[11] This is because the over-all average conceals underlying differences. Among the poorest members of society the actual population growth rate may be well below the *potential* growth rate because of deaths caused by poverty and malnutrition. At the same time the average product per person may be rising rapidly among the richer members. As a result the over-all average income per person will rise. However, it is important to note that

[10] See Boulding, K. E. (1966). The economics of the coming spaceship earth. In *Environmental quality in a growing economy, essays from the Sixth RFF Forum* (ed. H. Jarrett). Johns Hopkins, Baltimore.

[11] The potential growth rate is the rate at which population would increase if mortality rates were not influenced by low real income levels, particularly low nutritional levels.

so long as a case I situation persists it is not possible to secure a permanent, significant improvement in income among the poorest, numerically important group in the population.

Two other points may be noted in this context. The first is that the marginal product per person employed may not change at the same rate as the average product. For instance labour-saving innovation will cause the marginal product to rise more slowly or fall more rapidly than the average product. Thus there would be a tendency for the wage rate to fall in relation to other prices. For the working classes, labour-saving innovation could exacerbate the discomforts caused by a case I situation. The second point is that if the agricultural sector roughly coincides with the poorest, numerically important sector in a case I situation, there will be a growing tendency for food shortages to occur and therefore for food prices to rise.

In a case II situation there is an inherent tendency for living standards to rise. However, this may result in a relative decline in product prices and hence of the incomes of producers in certain sectors, possibly the agricultural sector. Hence a case II situation generally involves problems of adjustment of the sizes of the various sectors making up the whole economic system.

The size of the agricultural sector

What are the implications of changes in population and average *per capita* incomes for the relative size of the agricultural sector in terms of the number of persons employed? For purposes of this discussion let us assume that the 'agricultural sector' means that part of agriculture which is concerned with production for the home market. Crop and livestock production for export is therefore treated as part of the 'non-agricultural sector'. The need for this assumption does not arise, of course, when considering the whole world economy.

Changes in population and *per capita* incomes will cause shifts in both the supply of and the demand for agricultural produce. Note that the quantity of a particular product, wheat for instance, which is supplied over a given production period of a year depends upon the price of wheat. Hence the supply of wheat in a given year is defined by the *schedule* or *curve* relating the quantity supplied to the prevailing price. Certain problems of interpretation arise if this concept is applied simultaneously to the whole range of agricultural products. Nevertheless it is useful, for the purposes of this argument, to assume that a curve may be drawn relating the aggregate quantity of all agricultural products supplied to the mean price of these products. Such a curve is SS in Fig. 6.8.

It is drawn sloping upwards to the right, thereby implying that more of the produce would be supplied at a higher price. However, the slope is very steep, since the scope for varying the quantity in response to price changes in limited. Most costs are fixed within a single production period and the farmers' supply curves of effort may actually bend backwards as we have seen. Hence the supply of agricultural products is said to be inelastic.

Similarly, despite the conceptual difficulties, we can assume that a demand curve such as DD in Fig. 6.8 may be drawn, relating the aggregate

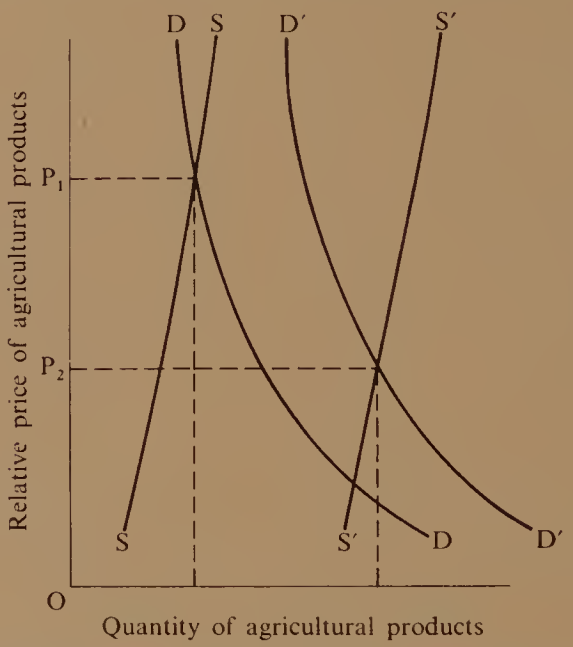


FIG. 6.8. Change in agricultural prices over time. The line DD represents the demand curve relating the quantity of agricultural products consumers are prepared to buy with the price at a given point in time. The supply curve SS relates the amount farmers are willing to sell to the price at a point in time. Over a given period both these curves are likely to shift to the right, say to positions D′D′ and S′S′ respectively. Since as incomes rise a decreasing proportion is spent on agricultural produce, the demand curve does not shift as rapidly as the supply curve. Hence agricultural prices tend to fall in relation to other prices (from P_1 to P_2 in the diagram).

quantity of all agricultural products demanded to the mean price of these products. This curve is shown sloping downwards from left to right since it is likely that less will be demanded as price rises. However, the downward slope is steep, showing that demand too is inelastic. Most agricultural products are necessities and there is little scope for varying consumption in response to changing prices. The equilibrium price at which

there is neither surplus nor deficit of agricultural produce is found where the curves cross, at price OP_1, in Fig. 6.8.[12]

Now if both the supply curve and the demand curve shift in the same direction at the same rate, say both supply and demand increase by 1 per cent per year, then the equilibrium price may remain reasonably stable. However, if, as is likely, the rates of change in supply and demand should get out of balance the equilibrium price will be effected. Shifts of the demand curve may result from (1) population change, (2) change in *per capita* incomes and their distribution, and (3) changing tastes and the development of substitute or complementary products. Shifts of the supply curve, on the other hand, are caused by (1) changes in the numbers employed in agriculture, resulting from population growth and movement into or out of agriculture, and (2) capital accumulation and technological change which for present purposes we may group together.

Let us consider a situation in which average income per head of population is rising. Note that this could be true of case I as well as case II situations. In either case it is implied that average product per head is generally rising. Initially let us assume that the proportion of the population engaged in agriculture remains constant, though we will find that this is unlikely to be true, and that average product per head is rising at the same rate in agriculture as in non-agriculture. In such circumstances it is likely that the supply of agricultural products will rise faster than the demand for them and therefore the equilibrium price will fall. This is because it is generally found that as income per head rises, the proportion spent on food falls.[13] The finding is so general that it has become known as a law, 'Engel's law', named after an early investigator of human consumption habits, Ernst Engel. Food makes up such a large proportion of total agricultural production that this law may safely be applied to all agricultural products. It follows then that the supply curve will shift further to the right over a given period, than will the demand curve as shown in Fig. 6.8 by the curves S′S′ and D′D′. Given these changes the equilibrium price falls to OP_2. Because both supply and demand are inelastic, a small discrepancy in growth rates will cause quite a large change in price.

[12] Note that throughout this discussion we assume that prices of agricultural products and *per capita* incomes are measured in *real* terms; that is in terms of the goods and services they will buy. Thus the effects of inflation on prices and incomes are excluded by definition. Some authors refer to the concept of the equilibrium, mean price level for agricultural products in real terms as 'the terms of trade for agriculture'.

[13] Another way of stating this law is that the income elasticity of demand for food is less than unity.

In the absence of government intervention it is likely that the actual price of agricultural produce will fall with the equilibrium price. This must mean a fall in the relative income of those employed in the agricultural sector, since we have assumed that the average physical product per person is rising at the same rate in all sectors. However, we must now relax the assumption that the proportion of the population which is engaged in agriculture remains constant, since the relative fall in agricultural incomes constitutes a clear incentive for people to move out of agriculture into other occupations. Migration from agriculture will usually have the joint effects of (1) reducing the rate of growth in the total supply of agricultural products and hence limiting the fall in agricultural prices and (2) increasing the marginal and average product per person remaining in agriculture. Thus an appropriate rate of migration out of agriculture could entirely eliminate the relative decline in agricultural incomes. In fact, if people are free to move between occupations we might expect the movement to continue until the marginal earning per man and the wage rate is equal in all occupations. If men could earn more by moving then they would move. This is the so-called 'principle of equal advantage'. Yet for various reasons differences in earnings and wages, particularly between agriculture and non-agriculture, may persist.

First, there are various elements in the total reward to labour apart from receipts in cash or kind. Receipts in kind will include free or cheap food and housing, for instance. In addition there are the fringe benefits of holidays with pay, pension schemes, and so on. Then there are non-pecuniary advantages or disadvantages—prestige, pleasantness of work, burden of responsibility, job satisfaction, working environment, social amenities, and so on. Although the working conditions may be more pleasant in agriculture than in many other industries, social amenities are frequently worse in rural areas than in the towns, so it is difficult to generalize about where the balance of advantage lies. In either case it is the total reward which a man takes into account in deciding where to work, and not just the pecuniary advantages. It should be noted too that the value of cash receipts may vary according to location if prices of consumption goods also vary. Thus if in urban areas goods cost on average 10 per cent more than in rural areas, urban money wages would need to be 10 per cent higher in order to provide the same level of living. In that case the *real* wages would be identical in the two localities.

Secondly, the existence of an appreciable degree of industrial unemployment may act as a constraint on migration out of agriculture. Unemployment may occur for various reasons, including the structural

unemployment caused by a failure of the non-agricultural sector to expand rapidly enough to absorb the labour movement out of agriculture. It may be argued that the alternative expected reward that a man takes into account in deciding whether to leave agriculture is the weighted average of the non-agricultural wage rate and the rate of unemployment benefit, if any. But for those who are risk averse the choice is different. Thus such a man is unlikely to give up a relatively secure income, albeit a relatively low one, in agriculture if the probability of becoming unemployed is high. Furthermore, if the level of unemployment is rising he will be less prepared to leave agriculture than if it is falling. At the same time general lack of information about opportunities in non-agriculture adds to the risk of moving.

Thirdly, and possibly contributing to the level of unemployment, there may be real barriers to migration, for instance when industrial trade unions operate closed shops or national governments of industrialized countries restrict immigration. Another form of effective barrier is the capital cost of changing occupations. This might be substantial if the migration involves moving home, say from a remote rural area to an urban centre, or if some kind of formal training is needed. Clearly if he is unable to raise the necessary finance a potential migrant will be unable to move; but even if he can raise the money the move will not be justified unless the gain in his total annual reward represents an acceptable return on the capital cost of moving.

Fourthly, the agricultural labour force is not homogeneous; members vary in their mobility. For instance a distinction may be drawn between farmers and hired workers. Generally speaking, farmers and their families have a stronger commitment to life on the land than their employees and are therefore less willing to migrate. Incidentally, a change from regular to casual employment in agriculture or from full-time to part-time farming represents partial migration out of agriculture. Then again if the young and well educated leave agriculture they have better employment prospects than their older and less well-qualified workmates. In fact the proportion and the absolute numbers of people in different occupations change, not just because individuals change their jobs, but because there are differing rates of entry by birth or by recruitment and exit by death or retirement. Thus one of the most important causes of the falling numbers employed in agriculture is the low recruitment rate of new, young members. As a result the average age of the agricultural work force gradually rises and the mobility of its members declines.

Finally, it should be apparent that migration out of agriculture is a gradual process; there is a time lag in the response to wage differentials.

But changes in the supply of, and the demand for, agricultural products are continuously occurring. The situation is a dynamic, moving one. Hence the static equilibrium state of 'equal advantage' is never achieved. Before the response to one price situation is complete prices will have changed again. Thus there may be a lasting tendency for the relative prices of agricultural products to fall, thereby causing agricultural incomes to remain below those in other sectors despite the continuing migration out of agriculture.

This, then, is the essence of the central problem of agricultural policy in most countries where average *per capita* incomes are rising. Without government intervention, agricultural incomes tend to fall below those of the rest of the community and the rate of migration out of the industry is rarely sufficient to prevent this. Hence agriculture may become one of the poorest, numerically important sectors of society. Government intervention to raise agricultural incomes may be called for on the grounds of social justice. Possible policy measures include:

(1) grants, training, and other assistance to encourage migration out of agriculture;
(2) direct relief and social security payments to farmers and rural workers;
(3) restricting agricultural supply by imposing production quotas;
(4) promoting demand by advertising, distributing free food to specific members of the community, and so on;
(5) subsidies on food and other agricultural products;
(6) fixing guaranteed prices above the equilibrium prices and dumping or destroying the resultant surpluses;
(7) fixing a guaranteed minimum agricultural wage rate.

I do not intend to elaborate on these alternatives here, but three general comments may be made. First, all these measures involve some kind of cost to society at large which must somehow be weighed against the benefits. Secondly, all except the first are likely to discourage migration out of agriculture. Now it may be that there is a strategic argument for limiting migration out of agriculture in that future events could radically alter the situation, causing food shortages and rising food prices. The populace might live to regret having allowed rapid migration out of agriculture. It is also claimed that the farming population provide a social service—for which they are not paid directly—in keeping the countryside attractive. Income support is needed to persuade them to remain and perform this useful function. Nevertheless policies which discourage migration out of agriculture must perpetuate the problem of

low farm incomes. Finally, many of the policies listed will yield greater benefit to large-scale prosperous farmers than to the smaller farmers who are most in need of help. Hence they are relatively inefficient ways of redistributing incomes to the needy.

Summary

1. Labour inputs are generally measured in man-hours, -days, or -years, regardless of variations in work output per hour. Inputs of regular family or hired labour are indivisible; they cannot be employed by the hour.

2. The seasonality of farm work causes seasonal variations in the opportunity cost of labour. Therefore it is not possible to ensure equality of marginal products in all uses at all times of the year.

3. Work peaks may be reduced by (a) omitting certain operations, (b) extending the time period over which operations are spread, or (c) increasing the rate of working by mechanization, improvement of working methods (work study), or by incentive payments.

4. Man derives satisfaction from his leisure as well as from his earnings. Thus we cannot predict how an individual will react to a rise in hourly income; he may choose to work more hours or to take more leisure. The offer of overtime rates will almost certainly persuade him to work more hours.

5. If population growth is such that average product per man tends to fall (case I situation) improvement in the real incomes of the poorest, numerically important sector is impossible.

6. When average product per man is rising (case II situation) or when average incomes are rising despite a case I situation, the agricultural sector is likely to decline in relative size.

Suggestions for further reading

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7. Capital

Basic concepts

Capital is made up of items which, unlike natural resources, have been produced by human activity but which have not yet been used up. For instance, a tractor is an item of capital; it has been produced but up until the time when it is finally scrapped it is not used up. Capital includes not only machines and tools but also buildings, roads, footpaths, ditches, fences, growing crops, livestock and stocks of food, seed, fertilizers, and other materials. It should be clear that some capital is needed for any kind of productive activity. Thus the spears and food and water containers of pre-agricultural, food-gathering societies are items of capital and so too are their food reserves. As an eminent economist has pointed out 'when Adam delved and Eve span there were evidently a spade and a spindle already in existence'.[1]

Each item of capital is known as an asset. We are concerned here with the assets used in the process of agricultural production, but two other forms of capital should be noted in passing. One is social overhead capital, consisting of the roads, railways, seaports, market places, research stations, advisory services, and so on, which yield benefits to the individual farmer but the provision of which is largely beyond his control. Thus, from the point of view of managing the farm, social overhead capital is best considered as part of the environment along with natural resources.[2] The other form of capital to be noted is consumer capital, made up of durable consumer goods such as houses, motor cars, furniture, and household appliances. These assets provide a stream of services for direct consumption, rather than contributing to the process of production. However, the distinction is not always clear, particularly in family farming where there are close links between farm and household.

[1] Robinson, J. (1953). The production function and the theory of capital. *Rev. Econ. Stud.* **21**, 81–106.

[2] The techniques of social cost–benefit analysis, used for appraising the provision of items of social overhead capital from the point of view of society as a whole, have much in common with the methods of capital planning discussed in this chapter. See for instance Mishan, E. J. (1971). *Cost–benefit analysis.* George Allen and Unwin, London.

For instance there may be difficulty in classifying the farmer's car or his bicycle.

Capital and money are often confused in everyday speech, but from an economic point of view it is important to distinguish between them because although most physical assets are bought and sold and can be evaluated in money terms it is the services provided by physical assets and not their money values which contribute to the process of production as capital. In order to emphasize this distinction in our initial example we will measure capital in physical units.

Consider a farmer who owns a flock of 100 sheep. It would simplify the

TABLE 7.1

Return on capital for sheep flock

Capital stock		Sheep
	Fixed (durable) capital	84
	Working capital	16
	Total	100
Annual returns and costs		
	Gross return	52
less	current operating cost (working capital)	−16
less	depreciation	−21
equals	net annual return	15

Rate of return = $\frac{15}{100}$ = 15 per cent

discussion if we could assume that all sheep were identical, i.e. the flock was homogeneous, but this assumption is likely to stretch the imagination too far. Instead we will assume that the flock consists of certain fixed proportions of males and females of different age groups. Additions to, and subtractions from, the flock are such as to keep the sex and age composition of the flock and hence the average value per sheep constant. (Note that even in this very simple example it is practically impossible to avoid comparing values.) Now assume that the farmer pays 16 sheep, in advance, to a shepherd in return for grazing and tending the flock over the coming year. During the year the remaining sheep bear 52 lambs, 21 of which are needed to replace old sheep which die, so that at the end of the year the owner is left with $100 - 16 + 52 - 21 = 115$ sheep. These results are also set out in Table 7.1. Note that we are in effect ignoring the

time it takes for lambs to grow into sheep, so that lambs can be assumed equivalent in value to sheep.

Three important concepts are illustrated by this simple example. First there is the concept of circulating or working capital. This is capital which is used up and replaced within a single production cycle of one year. Because of the seasonality of agricultural production there is often a delay of some months between the time when costs are incurred and the time when the product is obtained. Therefore these costs must be met out of reserves accumulated from past production, i.e. out of the stock of capital. In the present example we assume that the shepherd is paid for his labour services and for the land used before any product is obtained. He receives 16 out of the original stock of 100 sheep, leaving 84 to breed from. Thus the use of other productive resources, land and labour, necessarily involves the use of capital.[3] Note that in this example we are making the convenient assumption that the shepherd is willing to be paid sheep in return for his services. He in turn would exchange these sheep for clothing, food, and shelter for himself and foods, sprays, medicaments, and so forth for the sheep. Clearly in practice it is most convenient to hold working capital in the form of money since this is more readily exchangeable or 'liquid' than physical assets. Nonetheless it should still be remembered that the capital really consists of the stocks of clothing, food, shelter and so on which have been produced in the past.[4]

The second important concept is that of depreciation, which applies to durable assets such as breeding stock, perennial crops, improvements to land, buildings, and machinery. Although such durable assets are not used up within a single production cycle of one year they will not in general last for ever. Thus animals and trees eventually die, buildings deteriorate, and machines are broken up for scrap. Hence, if a farmer's operations are to continue on the same level he needs to replace his capital assets as they become worn out. In other words a recurrent replacement cost is associated with the use of any productive capital, although this

[3] For this reason the analysis of earlier chapters needs modification. It is not possible to vary inputs of labour or land while the quantity of capital is held constant; the employment of an additional unit of labour necessitates the use of a certain amount of working capital. Nevertheless, if inputs of labour or land together with associated working capital are increased in relation to other factors including other forms of capital, marginal returns are likely to diminish. The economic optimum rate of land or labour input must be redefined to take account of working capital (see footnote 6, p. 273).

[4] On the subject of working capital see Harrison, A. (1956). The 'capital profile' as an aid to decision making in farm management. *J. agric. Econ.* **12**, 64 and McFarquhar, A. M. M. (1961). The marginal value product of circulating capital in farming—implications of British and New Zealand cases. *Fm Econ.* **9**, 574–96.

need not arise every year. Thus if the productive life of a sheep is 4 years, the replacement cost will occur only once every 4 years. Note that the farmer pays out 16 sheep to the shepherd every year so these must be replaced annually. However for the remaining 84, which represent the fixed or durable capital, it is clearly more convenient to replace a quarter of them, i.e. 21 sheep, every year. However, even if for some reason it was necessary to replace all the sheep together at 4-yearly intervals, it would be convenient to treat the replacement cost as an imaginary annual series of cost items equivalent in total burden to the 4-yearly replacement cost actually involved. This annual replacement cost is known as 'depreciation'. It can be estimated in several different ways and is obviously dependent upon the length of life of the asset, which in turn is influenced by the amount of care lavished on maintenance. For instance if by greater attention to the health of the sheep their productive lives could be extended to 5 years, only 17 replacements would be needed each year; i.e., the depreciation would be reduced from 25 per cent to 20 per cent. Whether it is more profitable to replace after 5 years rather than after 4 years will depend upon whether the extra maintenance cost plus the loss of average lamb yield is greater or less than the reduction in depreciation.

The third concept illustrated by this example is that of return on capital, which is the total annual gain or benefit derived from using the capital, less all the extra costs including depreciation. It is generally expressed as a percentage of the initial capital stock. Thus in our example the total annual gain is 52 sheep, the current operating cost 16 sheep, and depreciation 21 sheep, leaving a net return of 15 sheep. Since the original stock consisted of 100 sheep, the annual rate of return is 15 per cent. Expressed in this way, return on capital is a pure number independent of the units in which the capital stock and the net return are measured. However, it is clear that the capital stock and the net return must be measured in the same units in order to calculate one as a percentage of the other. Note that this rate of return is a measure of the *average* net productivity of 100 sheep, not the marginal product of one more sheep.

We now have two alternative ways of viewing a capital asset, either as a stock of past production, in our example 100 sheep already produced, or as a source of a stream of future incomes, in this case 15 sheep per year continuing indefinitely into the future.

Investment

The acquisition of new capital is known as 'investment', and it can take place only if consumption is forgone, i.e. if saving occurs. Thus in

any given period everything that is produced is either consumed (used up) or it is saved. If it is saved it may simply be hoarded for future consumption or as a reserve against risk or it may be used for current investment.[5] As we have seen, some investment is necessary to replace assets which have become worn out, i.e. to allow for depreciation. Thus the net investment or capital formation is the total investment minus depreciation. Of course it may be possible to draw upon and consume existing capital reserves, in which case net investment is negative and the owner is said to dis-save.

In our sheep flock example (see Table 7.2), of the 52 lambs born, 16 are

TABLE 7.2

Investment and capital formation

Annual production		Sheep
	Total product less current operating cost 52 −16 =	36
less	consumption	−8
equals	Gross investment	28
less	depreciation	−21
equals	Net investment or capital formation	7
	Initial capital stock	100
	Rate of capital formation $\frac{7}{100}$ = 7 per cent	

needed to replace those paid out in current costs of shepherding and grazing. For convenience we will assume that the shepherd consumes the 16 sheep he receives. Now assume that the owner of the sheep flock consumes 8 more in the course of the year. The remainder, 52 −16− −8 = 28 sheep, which are not consumed, are therefore saved. None are hoarded so all these 28 join the main flock as investment. Of these, 21 are needed to replace sheep lost from the flock by depreciation—in other words to maintain the existing capital stock. Hence the increase in the capital stock, known as net investment or capital formation, is 28 −21 = 7 sheep. The capital stock at the start of the next year

[5] The reader may meet different terminology in some literature. For example Lord Keynes, defined any increases in stocks, even when undesired and unproductive, as investment. With this terminology investment must be identical with actual realized saving. However it is convenient for us to distinguish productive investment from hoarding.

(production cycle) will then be 107 sheep. The average rates of return and depreciation may change as a result of this increase in flock size.

In this instance the link between saving and investment is obvious; the owner forgoes the consumption of the sheep added to the flock. Although less obvious, the link is still there when the capital investment differs in kind from the consumption goods produced. For example, if the farmer decides to build winter housing for his sheep in order to increase the future lamb crop, he needs to accumulate reserves of food and other consumption goods (by saving) which he can then live on while he constructs the building. In an exchange economy most products are sold and most investments are made by purchasing capital assets. Certain individuals specialize in the production of capital goods such as tractors or buildings while others produce food, clothing, and so on for current consumption. Nevertheless for each individual, and indeed for the community as a whole, total income in any time period is either spent on consumption, hoarded, invested, or used to repay past borrowing.

An individual may spend in excess of his current income by borrowing, in which case the lender has done the saving. In order to be persuaded to lend, the latter will normally expect to receive a price for the use of his capital, namely 'interest'. Thus where capital can be borrowed or hired, interest is the unit factor cost to be compared with the rate of return in deciding whether or not to invest. To illustrate this let us assume pure competition which implies that the farmer may borrow as much capital as he wishes at a constant rate of interest, say 10 per cent. If he can lend out his own savings at the same rate of interest this also represents the opportunity cost for his own capital. When the rate of interest exceeds the return he can earn on his own farm he will benefit by lending to others. Let us also assume that the quantity of capital in use can be varied as in the example of the sheep flock, in which case the rate of return obtained by using one more unit of capital may be referred to as the marginal rate of return. Clearly, if the marginal rate of return exceeds the rate of interest, profit will be increased by making the investment. The economic optimum rate of capital use will be reached when the marginal rate of return is equal to the rate of interest, as shown in Fig. 7.1 at OA. If we assume, as in the diagram, that there are diminishing marginal returns to increased use of capital, then clearly a rise in the interest rate will reduce the economic optimum rate of capital input, while a fall in the interest rate will increase it.

Three additional comments may be made regarding the above discussion and the illustration in Fig. 7.1. First we may note that the argument is not substantially altered if we drop the somewhat unrealistic

assumption of continuously variable capital inputs. If investments are chunky or indivisible we may refer to each one as a project. To maximize profits, projects should then be ranked for adoption in diminishing order of their rate of return. All projects that yield a return which is greater than the rate of interest should be adopted.

The second point to note is that inputs of capital are assumed to be varied while the quantities of land, labour, and other inputs are fixed. Hence the law of diminishing returns is likely to apply, so that the marginal rate of return on additional units of investment eventually declines. However, we may question this assumption on the grounds that additional land, labour, and other inputs may be hired when capital

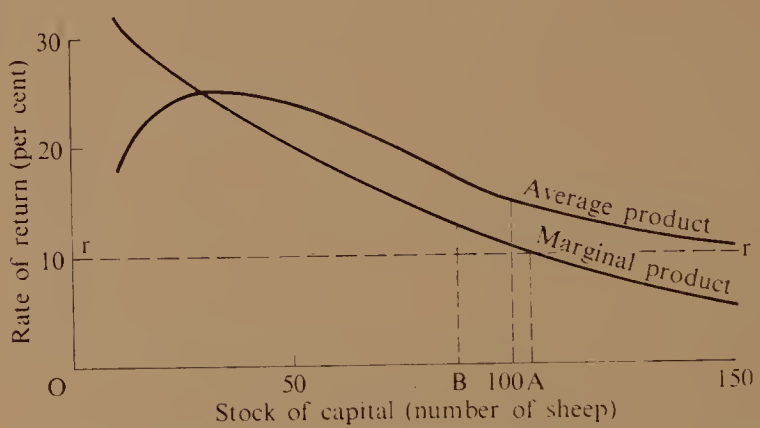


FIG. 7.1. The economic optimum stock of capital. Given an homogeneous form of capital measured in the same units as the product (e.g. sheep) one can envisage an economic optimum stock of capital (OA units of capital) at which the marginal rate of return on the last unit of capital is equal to the market rate of interest or opportunity cost. If the producer is risk averse he will employ a smaller stock (say OB units).

investment is increased. In our example a part of the capital stock of sheep is used to hire a shepherd and grazing. Thus if the use of capital is increased, all other inputs may be increased along with it so we have a situation of increasing scale rather than one of varying-factor proportions and there is no reason why marginal rate of return should decline. This question may be answered in two ways. On the one hand it is not always possible to hire additional land, labour, or management as and when required. One or other of these factors may be a constraint on production. Alternatively the price per unit of one or other of these inputs may rise in relation to the price per unit of capital. In either of these cases it will be profitable to substitute capital for the factor which has become a constraint or more costly; i.e. it may pay to vary factor proportions. Although

in many situations access to additional capital *may* enable production to be expanded to scale, at this stage we are concerned with the effect of varying capital in relation to other factors.

Finally it should be noted that the discussion so far has ignored risk and uncertainty. In fact, capital investment inevitably involves risk since there is a delay before the cost is recovered and the extra return is obtained. At the time of making an investment, a farmer cannot possibly know exactly what the outcome will be. In our example, when he decides to invest in 7 more sheep he is not sure what rate of return his flock will earn in the future.

Now, for a farmer who is indifferent to risk, the analysis does not need to be modified except that the costs and returns are not known with certainty; they are expected values. However, if, as is more usually the case, he has an aversion to risk he will be prepared to forgo some expected profit in exchange for a reduction in risk, as discussed in Chapter 4. This means that he is likely to use less than the previously defined economic optimum stock of capital so that the expected marginal rate of return will be greater than the interest rate as shown in Fig. 7.1 when OB units of capital are used. Note that the interest rate normally includes a risk premium to repay the lender for the risk *he* takes that his loan may not be repaid in full, but this does not allow for the borrower's risk.

Except where risk is specifically mentioned, in the remainder of this chapter we will make the simplifying assumptions that all estimates of costs and returns are expected values and that all farmers are indifferent to risk.

Compounding and discounting

So far we have avoided a central problem of capital theory: namely that of relating inputs supplied at one point in time to outputs obtained after a delay perhaps of some years. By assuming a balanced stock of capital we were able to treat inputs, depreciation, and outputs as constant from year to year so that current costs could be related to current returns. Furthermore by assuming that sheep produced were no different in kind from the sheep in the breeding flock we avoided all problems of capital valuation. Clearly these assumptions are quite unrealistic.

Consider, in contrast, a single investment project involving the purchase of a bunch of store cattle for £400 at a given point in time and their sale 2 years later for £625. For simplicity, we assume that no other costs are incurred. Such a simple case is known as a point-input, point-output process for obvious reasons. Now the costs and returns are measured in money terms. Note that in this case there is no depreciation; indeed, the

capital stock actually gains in value or 'appreciates'. Clearly an initial investment of £400 results in a total gain of £225 over 2 years but it is not immediately obvious how the total gain should be apportioned over the 2 years or how the capital stock should be evaluated in order to calculate the annual rate of return. Such an investment may be appraised by taking interest into account as a cost of production.

Thus if we assume that the farmer borrows the money to buy the cattle at 10 per cent interest, we can calculate how much he will owe at the end of 2 years. The calculations are set out in Table 7.3. In the first year the

TABLE 7.3

Compounding an investment (all values in £s)

	Year 1	Year 2	Total interest
Debt at start of year	400	440	
Interest on debt (at 10%)	40	44	84
Debt at end of year	440	484	

Thus £400 compounded at 10 per cent (0·10) for 2 years
$= £400 \times 1{\cdot}10^2 = £400 \times 1{\cdot}21 = £484.$
Terminal net worth = £625 −£484 = £141.

interest charge is 10 per cent, or 0·10, of £400 which is £40. Since the farmer has not yet received any returns from his project he must borrow this too so his total debt is increased to £440, that is £400 × 1·10. In the second year the interest charge is 10 per cent of £440 which is £44. By the end of 2 years the total debt will be £484, which is £440 × 1·10 or £400 × $1{\cdot}10^2$. Interest calculated in this way is known as 'compound interest' and the process of adding compound interest to an original sum of money is known as 'compounding'.

On the basis of this calculation it would appear that the proposed investment is likely to be profitable since the expected return of £625 is considerably greater than the total cost including interest. The terminal net worth, which is the investor's expected profit at the end of the project, is

$$£625 - £484 = £141.$$

Naturally if the interest rate should rise, so too would the total cost so that the terminal net worth would fall.

Any sum of money may be compounded at a given rate of interest over

a given period of time in this way to arrive at its 'future value'. For convenience Table A.1 has been included at the end of the book (p. 342), showing the future value of £1 compounded over different periods at different rates of interest. Any sum may be multiplied by the appropriate figure from the table to give its future value. Thus the future value of £400 compounded at 20 per cent, over two years is equal to

$$£400 \times 1{\cdot}44 = £576.$$

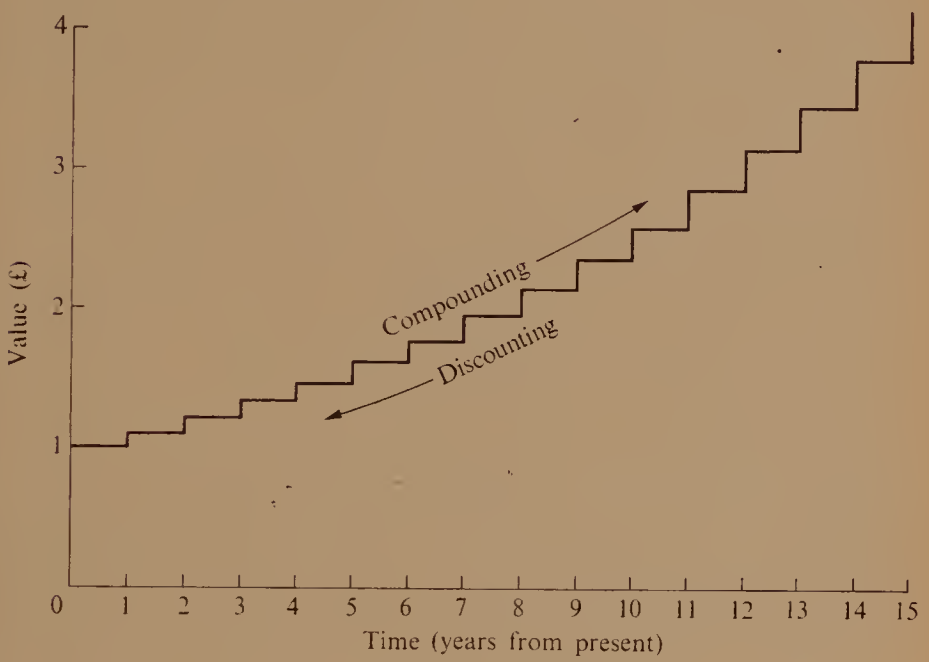


FIG. 7.2. Discounting: the converse of compounding. If an initial sum (principal) of £1 with compound interest at 10 per cent amounts to £4.18 at the end of 15 years, then by the same token a sum of £4.18 expected at the end of 15 years may be discounted at 10 per cent to give a present value of £1.

At this rate of interest the terminal net worth of our project is reduced to

$$£625 - £576 = £49.$$

Nevertheless, since this is pure gain, the investment is still worth making. Indeed, provided that there are no restrictions on the supply of capital and no risk, profits will be maximized by adopting any project for which the terminal net worth is greater than zero.

Although the estimation of the terminal net worth is a perfectly satisfactory method of investment appraisal in many circumstances, producers are generally concerned with the profitability of investments

as of now, not at some date in the future. Thus, for some purposes it may be convenient to think in terms of 'present values'. These may be arrived at by the technique of 'discounting' which is the converse of compounding (see Fig. 7.2).

To understand the logical basis of discounting we may note that, since interest is paid on borrowed money, £1 now is worth more than £1 next year, or at any time in the future. This is quite apart from the possible effects of inflation, which are difficult to predict and are therefore often ignored in investment planning. The money rate of interest may be influenced by the rate of inflation, however, and strictly speaking this should be taken into account. Either future costs and returns should be valued at forecast rates of inflation or, if they are evaluated at current prices, the 'real' (non-monetary) rate of interest should be used in compounding or discounting. The real rate of interest is approximately equal to the money rate of interest minus the rate of inflation.

A given sum of money is worth more now than at some date in the future simply because it can earn interest. For instance at a 10 per cent interest rate £1 now is worth £1.10 next year. Hence £1.10 earned next year is worth only £1 now, and £1 earned next year is worth only £1/1·10 = £0.9091 now. The remainder £1 − £0.9091 = £0.0909 is the interest charge, namely 10 per cent of £0.9091.

By the same token, if the interest rate is assumed to remain constant at 10 per cent, £1 expected in 2 years' time is equivalent in value to £0.9091 expected in 1 year's time which in turn has a present value of only

$$\frac{\text{£0.9091}}{1{\cdot}10} = \frac{\text{£1}}{(1{\cdot}10)^2} = \frac{\text{£1}}{1{\cdot}21} = \text{£0.8264}$$

Thus by repeated applications of the discount factor 1/1·10 we can arrive at the present value of £1, expected at any time in the future. The calculation is simplified by making use of discount tables such as Table A.2 at the end of this book (p. 343). The present value of any sum of money is found by multiplying that sum by the present value of £1 at the appropriate rate of interest over the appropriate number of years.

In our example the revenue expected at the end of 2 years is £625, so its present value at 10 per cent interest rate is:

$$\text{£625} \times 0{\cdot}8264 = \text{£516.50}.$$

Since this is larger than the initial investment of £400, the project appears profitable. Thus any point-input, point-output project may be assessed by comparing the discounted revenue with the initial investment.

Alternatively the 'net present value' (*NPV*) may be calculated by subtracting the initial investment from the discounted revenue

$$£516.50 - £400 = £116.50.$$

Note that £116.50 compounded over 2 years at 10 per cent amounts to £141, which is the terminal net worth of this project. The profit earned is the same whether revenue is discounted or cost is compounded.

Clearly the net present value of a project is dependent upon the interest rate. If the interest rate rises the present value of future receipts falls. Thus the present value of £625, in 2 years' time discounted at 20 per cent is only

$$£625 \times 0{\cdot}6944 = £434,$$

as compared with a present value at 10 per cent of £516.50. The net present value of the project is therefore reduced from £116.50 to £34.

Discounted cash flows

The example we have just considered—let us call it project A—is not very realistic. In real life most projects are likely to incur a series of input costs and may yield a series of receipts over time. Nevertheless the use of compounding or discounting can be justified on logical grounds. Now the series of annual costs or receipts (or the differences between them) may be referred to as 'cash flows', a cost being treated as a negative cash flow and a receipt being treated as a positive one. For this purpose annual costs do not include depreciation or interest charges. A conventional investment project will involve one or more negative cash flows followed by one or more positive cash flows.

Note that, since interest is charged annually on the debt outstanding at the beginning of the year, we have to treat all cash flows as though they occur on the last day of the year. In fact it is possible to calculate interest as a continuous charge (see Appendix p. 310) so that cash flows can be compounded or discounted from the precise point in time when they occur. However, this procedure is first more complicated and secondly less realistic since credit institutions actually do charge interest on an annual basis, at least on long-term loans.

The cash flows associated with an individual project are the *extra* costs and *extra* receipts associated with the project. Many farm costs, including those of existing buildings, machines, and other assets, are not affected by the new investment and may therefore be omitted from the calculations. However, if resources used in the project are limited in total and have alternative uses, then clearly their use should be charged

to the new project at their opportunity costs. In practice the estimation of opportunity costs may not be straightforward or easy, and for this reason whole farm planning may be preferable to the appraisal of individual projects. Nevertheless, provided that opportunity costs can be estimated, it is clearly simpler to appraise individual projects.

The costs of purchased inputs, including capital assets and working capital, are treated as negative cash flows in the years in which they are expected to occur. No depreciation charges are included, however, as this would represent double counting of the replacement costs of the assets. Naturally no interest charges are included as these are added in the compounding, or discounting, process. Expected revenues include the salvage value of each asset in the year it is to be sold. In so far as it is possible, grants and taxes should be taken into account.

To illustrate the compounding and discounting of cash flows let us

TABLE 7.4

Cash flows from two alternative projects (all values in £s)

	Initial	End of year 1	End of year 2
Project A	−400	0	+625
Project B	−400	+300	+286
Difference (A − B)	0	−300	+339

consider project B, the cash flows for which are set out below those for project A in Table 7.4. This could represent a similar bunch of cattle to that involved in project A but now some are sold after just one year. The total, undiscounted gain for project B is only £586 −£400 = £186 as compared with £625 −£400 = £225 from project A. However, there is an advantage in bringing some of the receipts forward to the end of year 1, in that total interest costs are reduced.

Again let us assume that the investment is to be financed by a loan on which interest is charged at 10 per cent annually. The results of compounding the cash flows are set out in Table 7.5, which may be compared with the results for project A given in Table 7.3. As before the initial £400 loan incurs an interest charge of £40 in the first year. The revenue of £300 at the end of the first year can be used to pay this interest charge and repay £260 of the initial loan. As a result the debt outstanding at the start of the second year is only £400 −£260 = £140. The interest charge on this is only £14, so the total interest charge is only £54 as opposed to £84 for project A. Nevertheless, at a 10 per cent interest rate the terminal

net worth is £132 which is less than the £141 for project A. Note however that at an interest rate of 20 per cent the terminal net worth of project B is £70 as against only £49 for project A.

Similarly we may discount the series of cash flows to give the present value. Again using an interest rate of 10 per cent the present value of future cash flows for project B is

$$(£286 \times 0{\cdot}8264) + (£300 \times 0{\cdot}9091) = £509.08.$$

Subtracting the initial investment of £400 leaves a net present value of £109.08 which as one would expect is equal to the discounted value of the terminal net worth

$$£132 \times 0{\cdot}8264 = £109.08.$$

TABLE 7.5

Compounded cash flows for project B (all values in £s)

	Year 1	Year 2	Total
Debt at start of year	400	140	
Cash flow	300	286	586
Interest on debt at 10 per cent	40	14	54
Capital repayment (cash flow less interest on debt)	260	272	532

Terminal net worth = £532 −£400 = £272 −£140 = £132

Thus the net present value (or the terminal net worth) can be calculated for any projected series of cash flows, at a given rate of interest. Incidentally, it is not necessary to use the same rate of interest throughout the discounting (or compounding) exercise. If the interest rate is expected to change at some date during the project life, then the subsequent cash flows are discounted to that date at the new rate of interest. The value of the project at that date is then discounted to the present, along with earlier cash flows, at the initial rate of interest. However, it is rarely possible to predict such changes in the interest rate and clearly the calculations are much simpler if a single rate is used throughout. The present value of future cash flows for a given asset is sometimes known as the 'capitalized value'.[6]

[6] Strictly speaking this analysis applies to all inputs including land and labour. Thus it is the *discounted* value of the marginal product per man which should be compared with the wage rate in deciding how many men to employ. Likewise it is the *discounted* value of the marginal product per hour which should be used in deciding how to allocate their time.

Now, if the farmer's objective is to maximize the total profit over a given, finite, period of time, he will achieve this by maximizing total net present value, or maximizing total terminal net worth which amounts to the same thing. Provided that there is no limitation on the amount of money that can be borrowed, or lent, at the given market rate of interest,

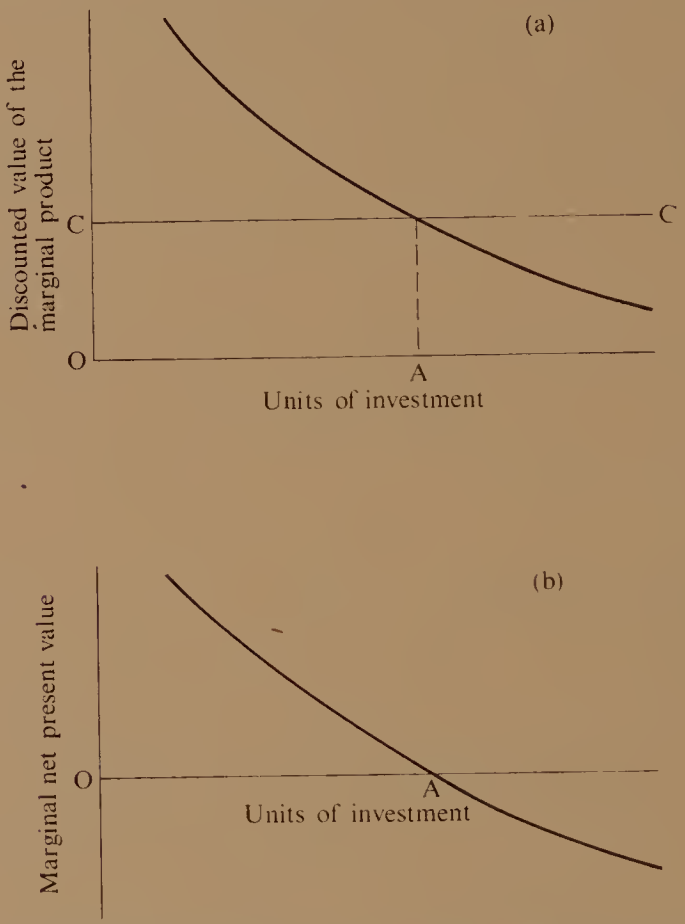


FIG. 7.3. Marginal product curves for capital investment. Given an homogeneous form of capital which can be measured in physical units and for each unit of which the discounted value of future returns can be estimated, we should be able to establish diminishing (discounted) marginal returns curves as shown here. The profit-maximizing level of investment OA is found either (a) where the discounted value of the marginal product is equal to the cost per unit of the capital investment or (b) where the marginal net present value falls to zero.

any investment which has a positive net present value will then be acceptable. When projects are mutually exclusive, as might be the case for projects A and B if the farmer only had sufficient housing for one bunch of cattle, they may be ranked for selection in order of their net

present values. On this basis, at an interest rate of 10 per cent project A is preferable to project B.

When the amount of money available for new investment is limited, the situation is more complicated, however. A possible approach would be to rank projects for selection in order of the present value *per* unit of capital investment, but with a variable pattern of cash flows over time, it is no longer clear how the quantity of capital should be measured. Also it does not allow for the fact that projects might be complementary to each other. These problems do not arise when we consider a specific kind of asset, sheep, or tractors for instance, provided that all units are alike or homogeneous. We may then construct a discounted value of the marginal product curve as in Fig. 7.3, assuming diminishing marginal returns to increased use of that kind of asset.

Strictly speaking, if the asset can be acquired only in chunky, indivisible units the discounted value of the marginal product will descend in a series of steps, but it is drawn as a smooth curve for convenience. The economic optimum number of units of investment is attained when the discounted value of the marginal product of one more unit is equal to or less than the purchase price, that is when no further increase in net present value is possible.

The internal rate of return

The net present value (or the terminal net worth) of a project is a measure of the total profit expected, when the cost of capital is taken into account. It does not measure the rate of return on capital. However, for many projects it is possible to estimate what is known as the 'internal rate of return' (*IRR*) even when the cash flows vary from year to year.

To illustrate let us again refer to project A. We found that if the interest rate used in discounting was increased from 10 per cent to 20 per cent, the net present value fell from £116.50 to £34. Now, if the interest rate is increased still further to 25 per cent the net present value falls to zero;

$$£625/(1{\cdot}25)^2 - £400 = £625 \times 0{\cdot}64 - £400 = £400 - £400 = 0.$$

Hence we could argue that the £400 invested in this project earns an annual rate of return of exactly 25 per cent. This represents the highest rate of interest the farmer could afford to pay, on funds borrowed to finance the project, without losing money. It is this rate which is known as the 'internal rate of return', or the 'yield' of the investment.

Generally, the simplest way to estimate the internal rate of return is by interpolation; that is by calculating the net present value for a range of

different interest rates and then plotting the results on a graph. Fig. 7.4 shows the relationship between net present value and interest rate for both projects A and B. The point where the curve cuts the rate of interest axis is the internal rate of return. For project A it is 25 per cent and for project B, 30 per cent. Now, provided that there is no limitation on the

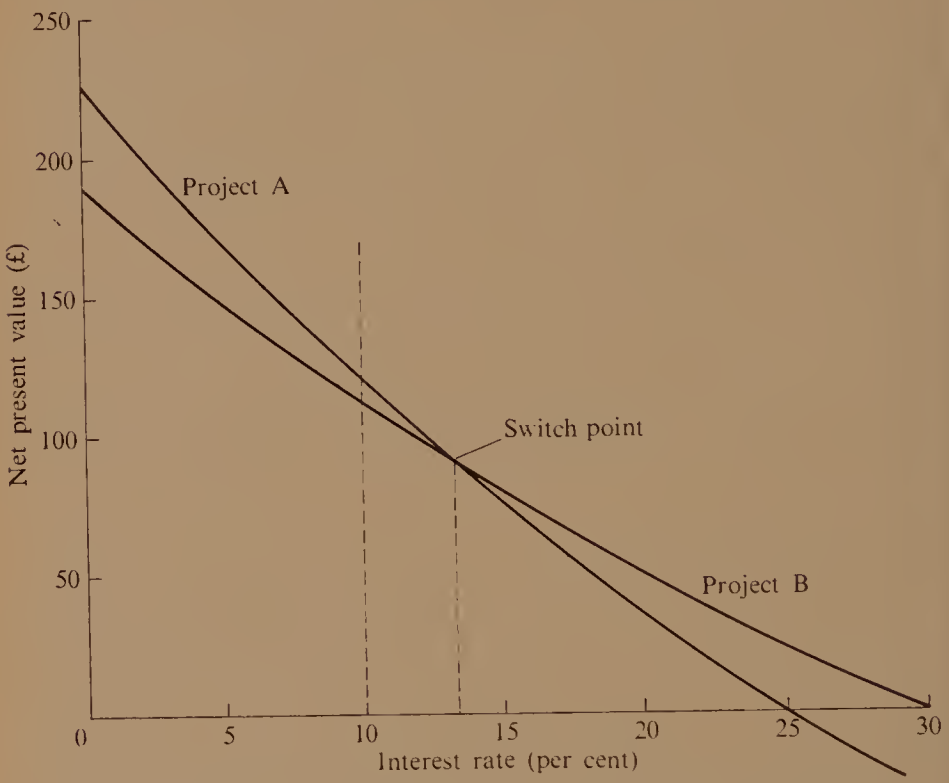


FIG. 7.4. Estimating the internal rate of return. For most normal investments the NPV declines as the interest rate is increased (for both project A and project B in this case). The interest rate at which the net present value is zero (for project A 25 per cent, and for project B 30 per cent) is the 'internal rate of return'. Note that in this case project A has the highest net present value at 10 per cent interest but then there is a switch at about 13 per cent beyond which project B has the higher value. Project B also has the higher internal rate of return.

amount of money that can be borrowed, at a given market rate of interest, profits will be maximized by adopting any project for which the internal rate of return exceeds the interest rate. Where this rule is applicable it will lead to the same selection of projects as the present value method. Thus at a 10 per cent interest rate both projects A and B are acceptable on both criteria.

However, where projects are mutually exclusive, so they must be ranked, the ranking based on internal rates of return may differ from that based on net present values as in the present example. At 10 per cent interest rate the net present value of project A exceeds that of project B but the internal rate of return for project A is less than that for project B. The reason why this occurs should be apparent from Fig. 7.4. Strictly speaking, although the same amount, £400, is invested initially in both projects, project A is a larger investment because there is a longer delay before revenues are obtained. Hence at relatively low rates of interest project A yields more profit or net present value. However, at interest rates above 13 per cent the extra cost of the larger investment exceeds the extra revenue and B becomes the more attractive project; the optimal choice switches from project A to project B.

In fact, when projects are mutually exclusive, the correct approach is to compare the 'incremental internal rate of return' with the rate of interest, the 'incremental internal rate of return' (*IIRR*) being the internal rate of return of the differences in cash flows between the two projects. It is also known as the 'marginal efficiency of capital'. The differences between project A and project B are given in Table 7.4. It should be apparent that with a cash flow of −£300 followed a year later by a cash flow of + £339, the incremental internal rate of return is 13 per cent:

$$£300 \times 1{\cdot}13 = £339.$$

Hence, provided that unlimited funds are available at 10 per cent interest, project A is more profitable than project B although the latter has a higher internal rate of return. If funds were limited, however, the incremental internal rate of return should be compared with the opportunity cost of capital, rather than the interest rate.

Unfortunately, the estimation of the internal rate of return is not always meaningful or even possible. No problem arises when we are dealing with a 'pure investment', such as projects A and B, in which costs (negative cash flows) are always followed by returns (positive cash flows), and never the converse. For such a project the net present value must fall as interest rate rises, so provided that total revenue exceeds total cost at zero interest rate, a unique and meaningful internal rate of return exists (see Fig. 7.4). If, however, costs occur *after* returns during the life of a project, it is no longer a 'pure investment'; for part of its life it is a dis-investment. The basic flaw in the concept of an internal rate of return in such circumstances is that it is not truly 'internal'; during the

periods of dis-investment the return must be earned *externally*, i.e. by investing the temporary surplus outside the project.[7]

It may be that relatively few independent investment projects are likely to involve negative cash flows towards the end of their lives, although there are examples such as the cost of grubbing out an orchard once it has stopped bearing fruit. From the point of view of conservation of natural resources, it is desirable that once a project is completed, the land should be returned to its previous state. However, the problem of

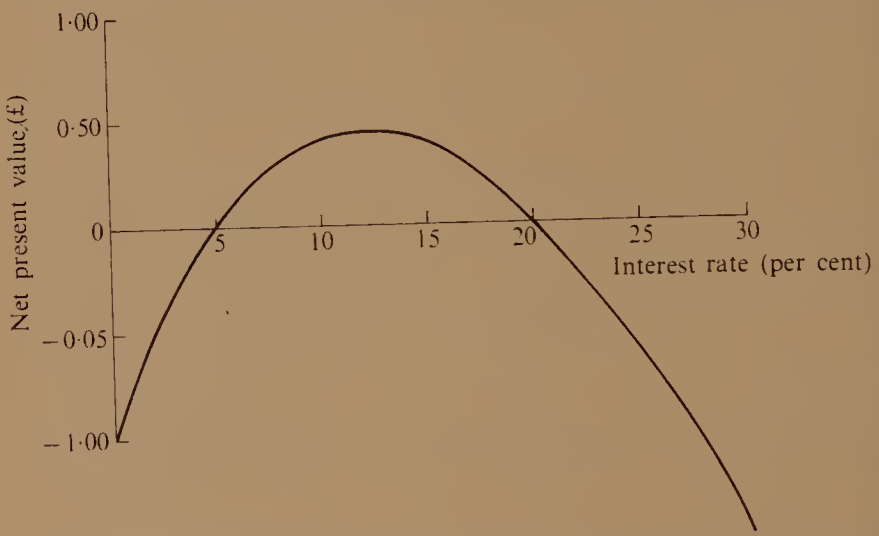


FIG. 7.5. Multiple internal rates of return. This shows the relationship between net present value and the interest rate for the differences in cash flows between project C and project B. (Note that it could also apply to a single investment project with negative cash flows at the end of its life.) There are 2 different internal rates of return and double-switching between projects C and B in terms of their net present values. Project B has the higher value at interest rates below 5 per cent and above 20 per cent but between these rates project C has the higher value.

'negative capital' as it is sometimes called may occur in the calculation of incremental internal rates of return.

For example, consider an alternative, project C which is compared with project B in Table 7.6. The graph relating net present value to rate of interest for the difference between the two projects is shown in Fig. 7.5. This shows that there are two interest rates at which net present value is zero, namely 5 per cent and 20 per cent, so there is no unique internal rate

[7] The problem is analysed more fully by: Bailey, M. J. (1959). Formal criteria for investment decisions. *J. polit. Econ.* **67**, 476–88, and Teichrow, D., Robichek, A. A., and Montalbano, M. (1965). An analysis of criteria for investment and financing decisions under certainty. *Mgt Sci.* **12**, 151.

of return. Apart from such cases of multiple solutions, other projects involving 'negative capital' may have *no* solution.

Optimization over time

When planning an investment, decisions must be made regarding not only the size of the project but also its length of life. For a single, non-repeated project, the total profit or net present value is maximized by continuing production until there is no further profit to be gained. At this point, shown in Fig. 7.6 at OA, the extra revenue from prolonging the

TABLE 7.6

Cash flows involving 'negative capital' (all values in £s)

	Initial	End of year 1	End of year 2
Project C	−500	+525	+160
Project B	−400	+300	+286
Difference (C − B)	−100	+225	−126
Analysis of C − B	Year 1	Year 2	Total
Debt	−100	+115	
Cash flow	+225	−126	+99
Interest at 10 per cent	−10	+11.5	+1.5
Capital repayment	+215	−114.5	+100.5

Terminal net worth = £100.5 − £100 = £0.5

project is just equal to the extra cost, *at market* prices, of the resources used. Indeed this optimum corresponds with that shown in Fig. 7.3 at OA, given that the units of investment are now measured as units of time. Obviously, this conclusion depends upon the assumption that the annual cash flows are declining over time and eventually fall to zero. This seems to be a reasonable assumption in many cases, although well-maintained roadways, ditches, hedges, and buildings may virtually last for ever.

Now if there were no fixed factors of production, the above rule for profit maximization over time would apply to all projects. However, when certain resources are fixed, as is more often the case, prolonging the life of a project has an opportunity cost, namely the revenues that could be earned by replacing the project with another. Profit maximization over time then implies maximizing *average* profit per unit of time. If the *same* process is to be repeated, the optimum replacement interval can be shown graphically as in Fig. 7.6 at OT time periods. At this point the extra, or marginal, profit to be gained by extending the interval is just equal to the

average profit per unit of time, which is therefore at its maximum. Note that all the curves in Fig. 7.6 are profit, not product, curves. When total profit is maximized, marginal profit is zero, and when average profit is maximized, marginal profit is equal to average profit.

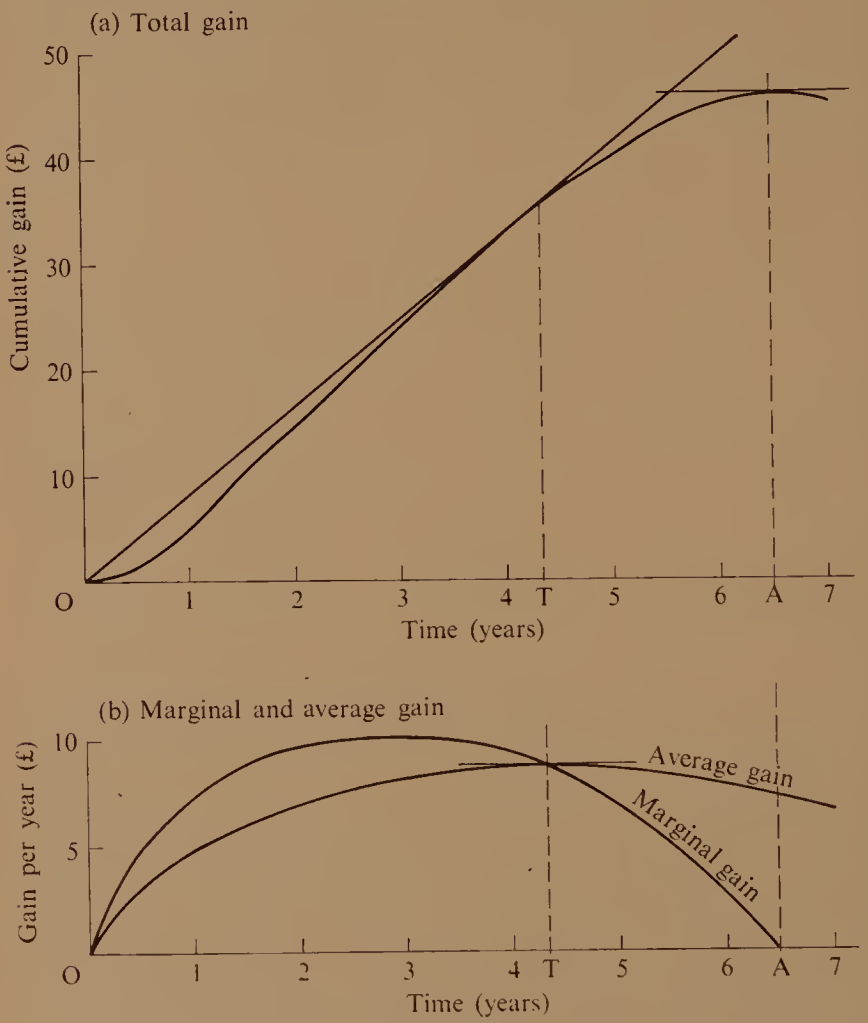


FIG. 7.6. Optimization over time. Profit or net gain is represented as a function of the length of the production period in the upper graph (a). The lower graph (b) shows the average and marginal net gains per year over the production period. Maximum profit for a single non-repeated project is found where the marginal gain from prolonging it is zero (i.e. at a project life of OA years). However, maximum profit per unit of time is found where marginal gain from prolonging the project is equal to the average gain per year (i.e. at a project life of OT years).

It may be helpful to illustrate the principles involved with a numerical example. Let us consider the problem of deciding how often to replace

ewes in the breeding flock. The hypothetical pattern of cash flows over the life of a typical ewe are set out in Table 7.7. Costs of labour and grazing are assumed to be fixed and are therefore omitted from the analysis. Now the objective is to maximize average annual gross margin which should lead to maximum average annual profit.

TABLE 7.7

Costs and returns over the life of a ewe

1 Year number	2 Costs (£)	3 Revenues (£)	4 Cash flow (£)	5 Cumulative total (£)	6 Resale price (£)	7 Total gross margin to date (£)	8 Average annual gross margin (£)	9 Increase in total margin (£)
Initial	9	—	−9	−9	9	0	—	
1	4	2	−2	−11	16	5	5	5
2	3	14	11	0	14	14	7	9
3	3	15	12	12	12	24	8	10
4	3	14	11	23	10	33	8.25	9
5	3	12	9	32	8	40	8	7
6	3	10	7	39	6	45	7.50	5
7	3	5	2	41	4	45	6.43	0

Notes.
1. The cash flow (column 4) is the difference between revenue (column 3) and cost (column 2).
2. Where the sign is omitted the number is positive.
3. The cumulative total (column 5) is the sum of the cash flows up to and including that year.
4. The total gross margin to date (column 7) is the cumulative total (column 5) plus the resale price (column 6). Thus it is the total gain that would be obtained if the ewe was sold in that year.
5. Average annual gross margin (column 8) is total gross margin (column 7) divided by the number of years elapsed (column 1).
6. Increase in total margin (column 9) is the difference between successive values of total gross margin to date.

From examination of Table 7.7 (and Fig. 7.6 which was derived from these data) it is apparent that, for a single ewe considered in isolation, the total gross margin is at a maximum when the animal is between 6 and 7 years old. However, for a sequence of ewes in a flock of a given size, the *average* gross margin per year is at a maximum when an animal is 4 years old, so this is the optimal replacement interval. At this age the increase in

total margin is equal to the annual average. From the data given in the table, it would appear that over a period of 12 years, replacement every 6 years would yield a total margin per ewe in the flock of

$$2 \times £45 = 12 \times £7.50 = £90,$$

whereas replacement every 4 years would yield a total gross margin per ewe of

$$3 \times £33 = 12 \times £8.25 = £99.$$

Now it will be noted that, in this analysis, we have taken no account of interest charges; we have not used discounting procedures. Strictly speaking we should have done. Rather than calculating total profit, or total gross margin, we should have calculated the net present value per ewe. However, it is not immediately obvious how the average annual return can be estimated from the net present value. Clearly, if we knew the average annual return and assumed that this was the actual amount earned every year of the life of the investment we could calculate the net present value. Such a stream of equal annual cash flows is known as an 'annuity' and Table A.3 at the end of the book (p. 344) gives the present values for annuities of £1 for different numbers of years, discounted at different rates of interest. As a simple illustration we will calculate the present value of an annuity of £8 per year over 5 years at 12 per cent interest rate. From Table A.3 (p. 344), the present value of an annuity of £1 for 5 years at 12 per cent interest is given as £3.6048, so for £8 per year the present value is

$$£8 \times 3{\cdot}6048 = £28.84.$$

If instead we are told that a particular project with a 5-year life has a net present value of £20 at 12 per cent interest, we can calculate the 'equivalent annuity' which over 5 years would yield the same net present value as follows:

$$£20/3{\cdot}6048 = £5.55.$$

Thus annuity tables may be used to convert the net present value of any iregular series of cash flows into an equivalent annuity. This equivalent annuity is, in effect, the average return over the life of the project, but it is an average which takes due account of time discounts.

We may now return to the data presented in Table 7.7 and calculate the equivalent annuities for each replacement interval, assuming in this instance that the market rate of interest is 12 per cent (see Table 7.8). As before, the optimal replacement interval is 4 years; at this age the net increase in total margin is equal to the equivalent annuity which is therefore

at its maximum. Note that the increase in total margin is now measured net of interest on the previous year's resale price. This is because we are now assuming that any such revenues can be invested to earn the market rate of interest. Delay in replacement involves this additional opportunity cost.

TABLE 7.8

Discounting and the equivalent annuity (at 12 per cent interest rate)

1 Year number	2 Cash flow (£)	3 Present value of cash flow (£)	4 Cumulative total (£)	5 Resale price (£)	6 Present value of resale price (£)	7 Total net present value (£)	8 Equivalent annuity (£)	9 Net increase in total margin (£)
Initial	−9	−9	−9	9	9	0	0	
								3.92
1	−2	−1.79	−10.79	16	14.29	3.50	3.92	
								7.08
2	11	8.77	−2.02	14	11.16	9.14	5.41	
								8.32
3	12	8.54	6.52	12	8.54	15.06	6.27	
								7.56
4	11	6.99	13.51	10	6.36	19.87	6.54	
								5.80
5	9	5.11	18.62	8	4.54	23.16	6.42	
								4.04
6	7	3.55	22.17	6	3.04	25.21	6.13	
								−0.72
7	2	0.90	23.07	4	1.81	24.88	5.45	

Notes.
1. Where the sign is omitted the number is positive.
2. Year number (column 1) cash flow (column 2) and resale price (column 5) are all taken directly from Table 7.7.
3. Present values of cash flows (column 3) and present values of resale prices (column 6) are obtained by discounting the corresponding figures for cash flows (column 2) and resale prices (column 5) respectively.
4. Cumulative total (column 4) is the sum of present values of cash flows up to and including that year.
5. Total net present value (column 7) is the sum of the cumulative total (column 4) and the present value of resale price (column 6).
6. Equivalent annuity (column 8) is total net present value (column 7) divided by the present value of an annuity of £1 for the same number of years and the same rate of interest.
7. Net increase in total margin (column 9) is the increase in total margin from Table 7.7 less the interest on the previous year's resale price.

For this example, the result obtained by using the more complicated equivalent annuity calculation is no different from that obtained by simple averaging. However the equivalent annuity is useful for comparing different, but competing, investments which have lives of different

lengths. Thus the above analysis is readily modified to take account of the possibility of introducing a new breed of ewe. Let us assume that the new breed yields a maximum equivalent annuity of £7 per ewe. On this basis it would appear profitable to change over to the new breed, since the maximum equivalent annuity given in Table 7.8 is only £6.54. However, this does not mean that the change should be made immediately. More profit can be earned by keeping the original breed of ewes until they are 4 years old, when the 'net increase in total margin' falls to £7.

Other replacement problems

1. *Orchard replacement*

An orchard is typical of longer-term investments for which the cost of establishment is spread over several years and the annual cash flow (margin over current costs) may be constant for many years after maturity. It is then only necessary to calculate the equivalent annuity for each year of the period over which yields and hence cash flows are

TABLE 7.9

Orchard replacement (at 9 per cent interest rate)

Age of trees (years)	Cash flow (margin) for that year (£)	Total margin to date (£)	Average Annual margin (£)	Present value of margin for that year (£)	Net present value (£)	Equivalent annuity (£)
50		2500	50		328.84	30
	55			0.68		
51		2555	50.10		329.52	30.03
	48			0.55		
52		2603	50.06		330.07	30.05
	40			0.42		
53		2643	49.87		330.49	30.06
	31			0.30		
54		2674	49.52		330.79	30.06
	20			0.16		
55		2694	48.98		330.95	30.05

declining with age. For illustration, the annual margins over current costs per hectare from the fiftieth to the fifty-fifth year of an imaginary apple orchard are presented in Table 7.9. The costs of grubbing trees out at the end of their productive lives are ignored in these calculations. However, these costs could readily be incorporated in the analysis by treating them as a negative resale (salvage) value.

In this case, if the trees are to be replaced with another planting of the same variety, then according to the criterion of maximizing average annual margin replanting should take place in year 51, whereas in order to maximize the equivalent annuity at 9 per cent interest replanting should be delayed until year 54. Thus the use of discounting techniques does alter the optimal frequency of replacement in this case. However, it is clear from the results given in the table that unless yields and cash flows fall drastically over a short period, the exact timing of replacement has little effect on the average margin *or* the equivalent annuity. If on the other hand yields and margins *do* fall drastically both criteria will give the same answer for the optimal time of replacement.

Having determined the maximum equivalent annuity this can be compared with the annual return from other competing activities with different lives such as production of annual crops.

2. *Tractor replacement*

Similar techniques may be applied to determine the frequency with which tractors or other machinery should be replaced in order to maximize profits. If we assume that the performance of the machine is not influenced by age, so that annual revenue is not affected by the replacement policy, the objective becomes that of minimizing annual cost. Again, strictly speaking, it is the equivalent annuity which should be minimized rather than the simple average cost. However as we have already noted whether the annual annuity or the average is used makes little difference to the optimal age of replacement. A hypothetical example is set out in Table 7.10.

The tractor is assumed to cost £1000 initially and the maintenance cost is assumed to increase from £50 in year 1 up to £250 in year 6. At the same time the resale value falls from £750 at 1 year old to £175 at 6 years old. On the basis of these figures, and assuming the tractor is to be replaced by a similar machine, it should be replaced every 5 years. If, however, as a result of technological developments a new machine becomes available, which will do the same work more cheaply, i.e. at a lower equivalent annuity cost, it may well be profitable to replace earlier.

3. *The point-input, point-output case*

Although it is highly unrealistic, the point-input, point-output type of investment is a simple and interesting case to consider. The optimal life is found when the annual percentage increase in sale price is equal to the interest rate. For instance, the cattle involved in project A are sold at 2 years old for £625. Now, if additional capital can be acquired at 10 per

cent interest, then it is worth keeping the cattle for a further year only if they are expected to gain in value more than 10 per cent of £625 which is £62.5. Thus if the cattle can be sold at 3 years old for, say, £690 it would be

TABLE 7.10

Minimizing the cost of tractor replacement

(a) Without discounting

Age of tractor (years)	Holding cost (purchase price plus maintenace) (£)	Second-hand value (£)	Cumulative cost less second-hand value (£)	Average annual cost (£)	Extra cost of delay (£)
0	1000	1000	—	—	
					300
1	1050	750	300	300	
					250
2	1100	550	550	275	
					250
3	1200	400	800	266	
					200
4	1300	300	1000	250	
					225
5	1450	225	1225	245	
					300
6	1700	175	1525	254	

(b) Discounting at 8 per cent interest rate

Age of tractor (years)	Discounted holding cost (£)	Discounted second-hand value (£)	Net present value of costs (£)	Equivalent annuity (£)	Extra cost of delay including 8% of second-hand value (£)
0	1000	1000	—	—	
					380
1	1046.30	694.44	351.86	380	
					310
2	1089.17	471.54	617.63	346.34	
					294
3	1168.55	317.53	851.02	330.22	
					232
4	1242.05	220.51	1021.54	308.43	
					249
5	1344.13	153.13	1191.00	298.29	
					318
6	1501.66	110.27	1391.39	300.97	

profitable to delay sale. If however the sale price at 3 years old is only £685 it is more profitable to sell after 2 years.

When finance is strictly limited, however, it has an opportunity cost which may exceed the market rate of interest. Since project A yields an internal rate of return of 25 per cent, under capital rationing, the opportunity cost of delaying sale is the 25 per cent that could be earned by repeating the project. Hence in order to justify selling at 3 years old rather than 2, the extra revenue would have to be at least 25 per cent of £625, which is £156.25.

The demand for capital

Throughout the foregoing discussion we have taken the view that capital is a factor of production; that along with land and labour it contributes services to the process of production. Furthermore we have assumed diminishing marginal returns (or diminishing incremental internal rate of return) as the stock of capital is increased in relation to the quantities of land and labour in use. From these premises we may infer, along the lines of the analysis of Chapter 3, that the equilibrium—or profit maximizing allocation of available capital within the farm business—is attained when the incremental internal rate of return per unit of capital is equal in all uses. That is to say the incremental internal rate of return for the last unit of capital used in root production is equal to, or as nearly as possible equal to, the incremental internal rate of return for the last unit of capital used in grain production and so on throughout the farm. At the same time, the equilibrium or economic optimum stock of capital is attained when this uniform incremental internal rate of return is equal to the interest rate, as illustrated in Fig. 7.1 or equivalently, in present value terms, in Fig. 7.3.

In addition, this analysis enables us to predict the likely influence of changes in prices or technology on the optimum stock of capital and hence on the rate of investment. For instance, we would expect a fall in the market rate of interest (i.e. the price of loans) to lead to an increase in the optimum stock of capital and thereby to stimulate new investment. Projects that were unprofitable at 10 per cent interest may become profitable if the interest rate falls to 8 per cent. For non-repeated projects for which the objective is to maximize total net present value, a fall in the interest rate means that future returns are discounted less heavily so the project life will be extended. On the other hand, for replacement chains as discussed above, increased investment as a result of a fall in interest rate implies *more frequent* replacement. Capital investment may also be stimulated by the following.

1. Technological innovations which increase the physical productivity of capital assets. The general effect of such innovations is to shift the marginal product curve—measured by either incremental internal rate of return or marginal net present value—upwards, and thus to increase the optimum stock of capital.
2. Increases in the product price in relation to the price of capital assets, which will have effects similar to those of technological innovations. In fact investment seems to depend far more on the product prices, or more precisely the expectations of future product prices, than on the rate of interest that must be paid for loans.

Difficulties in capital theory

Unfortunately, there are major difficulties associated with this so-called neo-classical theory of capital summarized above. Indeed some economists reject it completely as being 'spurious'.[8] Whilst we cannot deal with these difficulties in full it is necessary to be aware of them. Essentially they derive from the fact that capital has many dimensions; the quantity involved in a single project may well vary over time, time itself being another dimension. When one considers that the whole-farm business is made up of many overlapping projects it is not surprising that attempts to find a single measure of capital run into difficulty.

For purposes of exposition, simplifying assumptions are often made. Thus it is sometimes assumed that all capital is made up of a single homogeneous substance, or that it consists of a balanced stock of different kinds of capital combined in constant proportions, as we assumed at the beginning of this chapter. Clearly, having made such assumptions, there is no problem in finding a standard unit for measuring capital, but equally clearly the assumptions are quite unrealistic. Even for a single specific kind of asset such as a flock of breeding ewes, there are variations in age and quality between individuals; capital is just not homogeneous. At the same time, any change in the ratio of capital to labour involves a reorganization of methods of production and requires changes in the specifications of capital assets and the proportions in which they are combined. In a changing world farmers just do not maintain balanced stocks of different kinds of assets.

Thus we are thrown back upon the use of money values to quantify capital. Indeed the use of money values is necessary in any case, in order to relate the product output to the capital input in estimating rates of

[8] See, for instance, Robinson, J. (1953). The production function and the theory of capital. *Rev. econ. Stud.* **21**, 81–106.

return. However, this has the disadvantage that the values of existing assets are not constant but vary over time with changes in the market rate of interest, the product price, and the available technology.

A clear distinction should be drawn between the situation at the planning stage, *before* investments are made, when the available funds are highly liquid and can be allocated in any direction, and the situation *after* investments have been made, when flexibility has been lost. Thus new investments can be planned in such a way as to maximize expected net present value, but once funds are invested in a piggery, for instance, it will be difficult and costly to convert this into an orchard in response to changing prices. The metaphore often used is that of the relationship between putty and clay; before investments are made the capital, in the putty state, can be put to any use and shaped into any form, but once invested the capital is baked hard into clay.

Existing assets may be evaluated in three different ways: in terms of *either* the cost of producing them *or* their future productivity *or* their current market prices. The three approaches may yield quite different valuations for a given asset, at a given point in time. Note that the first, 'backward-looking', method involves compounding past costs, minus past revenues, up to the present, whereas the second, 'forward-looking', method entails discounting expected future revenues, minus future costs, back to the present. Generally speaking, the latter would seem to be the most appropriate measure of current value for most purposes. Hence the current market price might be expected to reflect the present value of future returns.[9] However the second-hand market for many assets is limited and the transaction costs are high. As implied above it is difficult and costly to dispose of a building such as a piggery. In many cases the scrap or salvage value of an asset is considerably less than the present value of expected returns.

At the economic optimum, or equilibrium, level of use of a particular kind of capital, the discounted future revenue for the last unit is just equal to its cost, as shown in Fig. 7.3. In other words, at the equilibrium point both the backward-looking and the forward-looking valuations give the same result. But during the life of an asset, changes may occur which cause the two valuations to differ. For instance a fall in the interest rate will cause future returns to be discounted less heavily, so that the present values of existing assets will normally tend to rise. However, if the falling interest rates are associated with rising wage rates, the net result

[9] Note that there is the additional problem, which applies to the evaluation of any goods or services, that the market price represents the value of the last (marginal) unit sold and may not necessarily apply to all units in existence.

could be a fall in future margins and hence a *fall* in the present values of existing assets.[10] Then again technological change will cause accelerated depreciation due to obsolescence. There will be 'fossils' in a farmer's stock of capital equipment resulting from erroneous past investment decisions. Once changes have occurred it is too late to reverse these earlier decisions. Thus, capital gains will be made on some investments and capital losses on others.

All this means that the rate of return on the original cost of a capital asset does not measure the productivity of that asset. Past costs are fixed or unavoidable and the earnings largely consist of pure profits, which may of course be negative. If on the other hand, assets are evaluated in terms of what they are expected to earn in the future, then any attempt to estimate the expected rate of return on this value involves an impossible circularity.

At the same time, it is clear that changes in the *value* of a farmer's capital stock are different from changes in the *quantity* of that stock. The value of capital is *not* an appropriate measure for assessing rates of accumulation and growth. How then should the total quantity of capital be measured? Whilst a definite answer cannot be given, three observations may be made. First, it is clear that we can and do make subjective assessments of the total quantity of capital. Probably most observers would agree that there is more capital per hectare and per man on a typical European farm than on a typical African farm. However, difficulties would arise in attempting to *measure* the difference. Secondly, index numbers have been devised as measures of the aggregate quantity of capital, adjusted to exclude the effects of price changes.[11] However, by definition an index is not a direct measure, so it must be interpreted with care. Thirdly, for many purposes aggregation is unnecessary. Thus accumulation and growth may be measured in terms of increases in the numbers of physical units of each individual class of assets. Problems then arise in allowing for age differences and technological changes, though these are not insoluble.

None of these problems would seem to be relevant to the planning of new investment, whether it is for replacement of depreciated items or to add to the capital stock. In this context, it is appropriate to measure the investible resources about to be committed in terms of their expected

[10] The first of these effects is known in the literature as the 'Wicksell effect' and the second as the 'reverse Wicksell effect'.

[11] See for instance Champernowne, D. G. (1953). The production function and the theory of capital: a comment. *Rev. econ. Stud.* **21**, 112. Also Boulding, K. (1950). *A reconstruction of economics*, Chapter 10. Wiley, New York; and Green, H. A. J. (1964). *Aggregation in economic analysis; an introductory survey*. Princeton University Press.

values. The total amount a farmer will plan to invest over a given time period is likely to be influenced by his expectations of interest rates and product and factor prices, in the manner suggested earlier. Of course mistakes will be made, and for this reason plans must be revised continuously over time. At any point in time the existing stock of tangible assets will influence the alternatives open to the decision maker, but their original costs are irrelevant to current decisions.

There is one final problem, which may in fact cast doubt on the validity of applying marginal productivity theory to new investment. This is the problem of 'double-switching' or 'capital reversing', which we met with in the form of multiple solutions for the internal rate of return calculation.

This was illustrated by the cash flows given in Table 7.6 for the difference between the projects B and C. The net present value of the difference between the two projects was found to be zero at 5 per cent *and* at 20 per cent rate of interest, as shown in Fig. 7.5. From this diagram we see that project B has a higher present value than project C at rates of interest above 20 per cent, in other words project B represents a larger capital investment. However, if the rate of interest should fall below 20 per cent, the ranking of the projects is reversed; there is a switch from project B to project C as the larger and more profitable investment. If the interest rate should fall still further, below the other 'switch point' of 5 per cent, then the ranking switches back and project B again becomes the larger and more profitable. This seems to make the whole concept of quantity of capital, or of investment—and hence of diminishing returns to increased investment—quite meaningless.

The practical importance of this problem depends upon the extent to which investment must be divided up into discrete projects. If capital investment could be varied smoothly and continuously the problem would not exist. However, even when investment can only be varied in discrete 'lumps', provided that there is a large enough number of substitutable projects to choose from, double-switching is unlikely to occur.

Savings and the rate of return

In the discussion so far, little thought has been given to the main sources of funds for new investment, namely savings and credit. Let us first consider the factors influencing the rate of saving. As was noted earlier, saving means postponing consumption: i.e. exchanging current consumption for future consumption. The extent to which a farmer, or any other individual, will choose to do this depends upon his pattern of 'time-preference'. Clearly patterns of time-preference vary from one

person to another but we may distinguish three factors likely to influence anyone's attitude to saving. These are:

(1) his expected average income;
(2) expected fluctuations in income; and
(3) the rate of return on investment.

Ability to save obviously depends upon a man's level of income. A smallholder who is barely managing to provide food, clothing, and shelter for his family—who is operating at subsistence level—cannot afford to save. Among those who are able to save, the rate tends to increase with increasing income. Indeed the 'marginal propensity to save', which is the *proportion* that is saved from each additional £1 (or other unit) of income, is generally found to rise as real incomes rise. Thus a farmer's rate of saving is directly influenced by his expected average income.

At the same time a farmer will vary the rate of saving over his lifetime so as to smooth out fluctuations in income. In particular he will save during his working life in order to provide for his retirement and possibly an inheritance for his heirs. A typical pattern is for a young man to borrow during the period of establishment of his farm and his family. From then on he saves, initially to repay his debts and then to provide for his old age. In retirement he lives on his past savings. Naturally this pattern of behaviour will vary to some extent according to the social system, although saving for old age seems necessary in any society. In addition, during periods of high income, savings may be made to provide a reserve as security against emergencies, an accident, a long illness, a crop failure, or a livestock disease outbreak.

The third main factor affecting the rate of saving is the rate of return on investment. Now it is commonly assumed that farmers, like most people, are 'impatient', or averse to waiting. They are supposed to have a positive 'rate of time-preference', which means that they will save only if they expect to derive some positive gain in the form of a return on capital investment. This assumption is clearly false: people would save to provide for their old age, whether or not they expected to earn any return on their savings. Nevertheless, it seems likely that the rate of return will have some effect on the level of saving and on the proportion of saving that is invested rather than hoarded. It is generally assumed that the higher the expected rate of return, the more a farmer will be induced to save and invest. This implies that his 'marginal rate of time-preference' (*MRTP*), also known as his 'personal discount rate', increases with increased saving and investment per unit of time as shown in Fig. 7.7.

Even this assumption is open to doubt because a rise in the rate of return on investment means that a given, target level of future income can be obtained from a smaller amount of current investment. A farmer might therefore decide to save and invest *less*, not more, as the rate of return rises. However, evidence suggests that this is not normally the case.

Now, assuming that double-switching is not a problem, the incremental internal rate of return will diminish with increasing investment per unit of time owing to the operation of the law of diminishing marginal returns. This is shown by the incremental internal rate of return curve in Fig. 7.7. The economic optimum rate of investment is found at A, where the

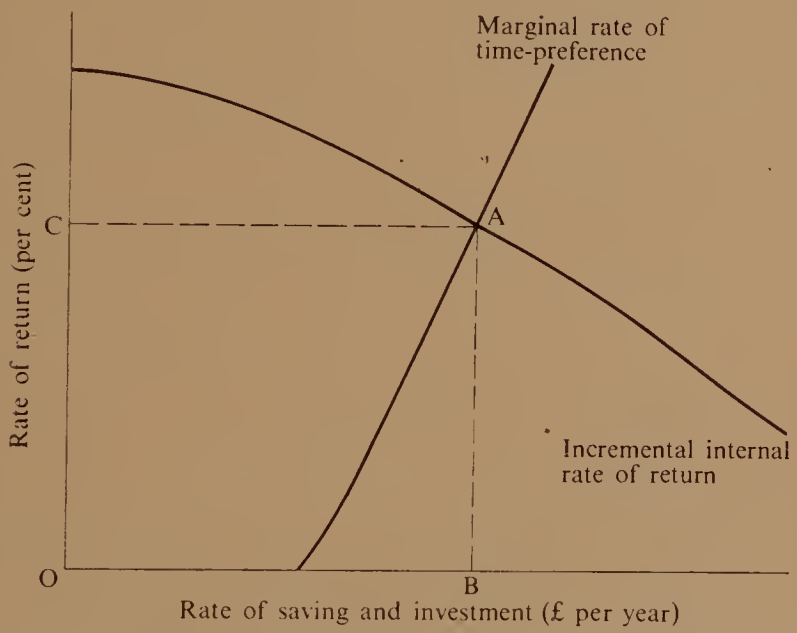


FIG. 7.7. The optimum rate of saving and investment. Given that double-switching is not a problem, the incremental internal rate of return is likely to diminish with increased annual investment as shown. At the same time there will be a limit on the amount an individual is able or willing to save, although the limit may increase if the rate of return does, as shown by the marginal rate of time-preference curve. The optimum rate of saving and investment is found where the two curves intersect at point A with £OB invested per year at an incremental rate of return of OC per cent.

incremental internal rate of return is just equal to the farmer's marginal rate of time preference at OC per cent. Thus for projects financed by the farmer himself the opportunity cost of capital is equal to his marginal rate of time-preference. Capital financed from savings is often called 'equity capital' to distinguish it from borrowed or 'debt capital'.

Naturally the optimum rate of investment and the opportunity cost

of equity capital will be different for different individuals and indeed for a single individual at different points in time. As saving and investment continue over time the *total* stock of capital is increased so, if some other resources are fixed, we might expect the incremental internal rate of return curve for new investment to shift downwards. Eventually this could lead to a situation where the incremental internal rate of return is so low that no new net investment is justified; a static optimum stock of capital will have been attained.

In practice the situation is complicated by the processes of change.

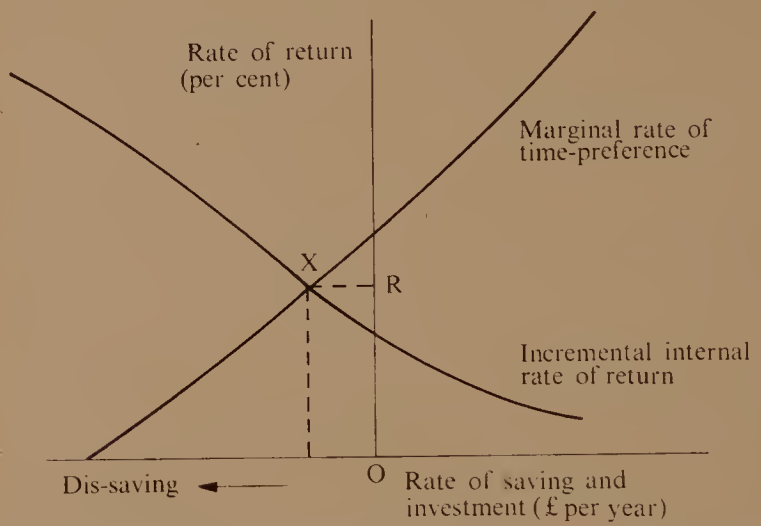


FIG. 7.8. The optimum rate of dis-saving. At certain stages of his life cycle, or under unfavourable economic conditions, a farmer may prefer to dis-save and dis-invest; his marginal rate of time-preference curve intersects with the incremental internal rate of return curve at a negative rate of investment, despite a positive rate of return of OR per cent.

Thus it is unlikely that quantities of other resources will remain fixed. The labour force may well decline, for instance, and this might affect the rate of return on investment. Linked with this is the effect of technological innovations which in general yield higher risk-discounted returns than do investments in existing activities. These effects may well offset the diminishing returns to increases in the total stock (as well as making measurement of the capital stock difficult) so that although the incremental internal rate of return curve will shift over time it will not necessarily fall continuously.

However, as we have seen, it is quite common for an individual's marginal rate of time-preference curve to shift backwards over his lifetime until at retirement he prefers to dis-save; his optimal rate of

investment is then negative even though the incremental internal rate of return curve and his marginal rate of time-preference curve intersect at a positive rate of return, as shown in Fig. 7.8.

Borrowing and risk

We assumed earlier, for simplicity, that the farmer can borrow, or lend, any amount of capital at a constant market rate of interest: i.e. we assumed pure competition in the capital market. On this assumption it is rational for the farmer to borrow, or lend, capital until his personal discount rate is equal to the market rate of interest. However, the assumption of pure competition in the capital market is not valid. For instance, it is generally found that the rate of interest which a farmer has to pay if he borrows capital is greater than the rate of interest he can earn by lending to others; the 'borrowing rate' exceeds the 'lending rate'. In such circumstances three different cases may be distinguished. First, where the incremental internal rate of return on the last unit of the farmer's own investment exceeds the borrowing rate, it will be profitable for him to borrow funds for investment until the incremental internal rate of return is equal to the borrowing rate. His opportunity cost of capital will then be measured by the borrowing rate. Secondly, where the incremental internal rate of return on his own savings is less than the lending rate, he will gain by lending funds to others and reducing his own investment until the incremental rate of return is equal to the lending rate. In this case the lending rate measures the opportunity cost of capital. Thirdly, where the incremental internal rate of return on his own funds lies between the borrowing and lending rates he should neither borrow nor lend. The opportunity cost of capital differs from both market rates of interest.

There is a major weakness in all these theories, namely that they do not take due account of risk aversion on the part of both borrowers and lenders. Let us first consider the effect of risk-aversion on the borrowing behaviour of farmers. Although it is generally recognized that risk aversion will cause farmers to invest less than the economic optimum amount (see p. 267), there is considerable doubt about the precise way in which risk aversion is, or should be, taken into account. One suggestion is that a risk premium should be added to the opportunity cost of capital. Thus if the farmer borrows capital at 10 per cent interest, then he may decide to adopt only those projects which are expected to earn at least, say, 15 per cent incremental internal rate of return; which means that he has added a risk premium of 5 percentage points to the cost of capital. Clearly this will reduce the number of investments that are judged

acceptable. The size of the risk premium is based on subjective judgement and reflects not only the variability of returns on investment but also the farmer's attitude to risk. The more cautious he is, the higher his risk premium will be and hence the smaller the amount of investment he will make. However a single individual may vary his risk premium according to the riskiness of the project under consideration.

This method of adjustment for risk in investment appraisal has the advantage of simplicity but it has little theoretical justification. A more meaningful approach would be to consider the probability distributions of the major costs and benefits of each project. However, the problem of choice is much more complex than in the single-period analysis discussed in Chapter 4. Now the decision maker has a whole series of expected cash flows and associated variances to take into account.[12] For the remainder of this discussion we will assume that some estimate is made of the probability distribution of rates of return, or net present values, and that the most pessimistic estimates are taken into account along with expected values in deciding how much to borrow and invest.

As we have already seen, risk aversion will cause a farmer to invest less than the economic optimum amount. When the most pessimistic estimates are taken into account, some projects which, on average, yield an incremental internal rate of return greater than the opportunity cost of capital will be rejected. This is true for both equity and debt capital. However, in practice more risk is associated with borrowing than with investments financed from savings. This is because the annual interest charges on borrowed money do not vary with year-to-year fluctuations in the rate of return on capital. As a result the variation in net returns after interest is paid is greater than the variation in the rate of return on all capital. This 'principle of increasing risk' is illustrated by the hypothetical figures given in Table 7.11. The 'equity ratio' is the proportion of total assets which are owned by the farmer. Alternatively the 'gearing' or 'leverage' is the ratio of borrowing to equity capital. In good years, when the average rate of return on capital is greater than the interest rate on borrowed funds, then the average rate of return on the farmer's own capital is increased by borrowing. But in poor years, when the average return on capital falls below the borrowing rate, the farmer with a low equity ratio makes losses.

Thus the principle of increasing risk sets a limit on the amount a

[12] The estimation of his expected utility function is of little help in this respect since it is likely to change over time, especially if his capital stock is changing. Thus a utility function estimated at one point in time is not appropriate for evaluating costs and benefits expected several years later.

farmer will be prepared to borrow over a given time period. In this example, if the farmer is unwilling to allow his income to fall below zero even in poor years, then he will restrict his borrowing to 50 per cent of his total assets, even though on average he could earn more by increased gearing. Clearly as he acquires more assets of his own he can afford to

TABLE 7.11

Borrowing and risk: the effect of gearing on the return on equity capital

	Equity ratio		
	100 per cent	50 per cent	25 per cent
Equity capital valuation (£1000s)	20	10	5
Borrowed capital used (£1000s)	0	10	15
Total (£1000s)	20	20	20
Good years (average return on capital 20 per cent)			
Total return per year (£1000s)	4	4	4
Interest on borrowed capital at 10 per cent (£1000s)	0	1	1·5
Residual (£1000s)	4	3	2·5
Average return on equity capital (per cent)	20	30	50
Poor years (average return on capital 5 per cent)			
Total return per year (£1000s)	1	1	1
Interest on borrowed capital at 10 per cent (£1000s)	0	1	1·5
Residual (£1000s)	1	0	−0·5
Average return on equity capital (per cent)	5	0	−10
Expected average rate of return on equity capital assuming good and poor years equiprobable (per cent)	12½	15	20

borrow more without decreasing his equity ratio and hence without increasing his risk.

Credit

Persons and organizations that lend to farmers must face the risk of loans not being repaid in full. This does not imply that farmers are dishonest but rather that they are fallible. They may make errors of judgement: their investment ventures may fail and all or part of the capital may be lost. Hence the credit agency, before lending to a farmer, will generally require some reassurance that the loan will be repaid in full.

The amount of credit made available to a particular farmer, the duration of the loan, and the interest rate charged upon it are likely to reflect the kind and degree of reassurance he can offer.

In some cases the borrower can provide the necessary assurance by offering some physical asset as collateral security for the loan. This means, in effect, that the credit agency becomes the legal owner of the asset with the right to dispose of it if the loan is not repaid within a specified period. Naturally once the debt is paid, ownership reverts to the borrower. The duration of such loans will depend upon the durability of the asset offered as security. For instance long-period loans can be offered against durable assets such as land and buildings, which are then said to be mortgaged. On the other hand, loans for hire purchase of machinery are repaid over a shorter period because of the more rapid depreciation of the security. Clearly, if the amount of debt outstanding is always less than the value of the asset offered as security the lender's risk is greatly reduced. In such circumstances credit agencies can offer loans on a formal contractual basis without needing much specific knowledge of local conditions or the borrower's competence and ability to repay.

By contrast, unsecured credit can be offered only where there is thorough knowledge of the particular case. Thus agricultural merchants are able to offer unsecured credit, albeit only for short periods, on the basis of personal knowledge of the area and of the farmer with whom they are dealing. In many countries, government-sponsored farm-credit schemes are administered through the agricultural extension service, since an extension worker is expected to have the necessary local knowledge to assess which farmers are credit-worthy. The extension agent may indeed guide and supervise the use of the credit which is provided.

Clearly the assumption of an unlimited supply of capital at a constant price (interest rate) is false. When a physical asset is offered as security for a loan it is never possible to borrow the full market value of that asset. The initial 'down-payment' must be found by the farmer himself. Furthermore many forms of capital, particularly working capital, are unsuitable for use as physical security for loans. In this context we may note that the more capital a farmer already owns, the more physical security he can offer and hence the more new funds he is likely to be able to borrow over a given time period. The amount a farmer will be able to borrow *without* offering physical security is likely to vary according to assessments of his personal qualities but it will nonetheless be limited in total.

Rather than there being a fixed limit on the amount a farmer can borrow over a given time period, extra funds may be available but at an

increasing rate of interest. The more a particular farmer borrows the greater is the risk that he will be unable to repay his debts and the smaller the proportion of his debts which can be secured against physical assets. Thus it is likely that in order to accept this increasing risk credit agencies will require to be paid a higher rate of interest; the risk premium increases. As a result the marginal cost, or rate of interest, curve for borrowed capital may be similar in form to the marginal rate of time-preference curve for personal savings shown in Fig. 7.7.

Multiperiod planning

It should now be apparent that the problem of planning the whole farm over time, i.e. of choosing the optimal growth path, is very complicated indeed. The techniques discussed earlier in this chapter for appraising individual projects are inadequate for solving this larger problem. Four complications merit particular attention.

1. *Capital rationing*

When the quantity of funds available for new investment at a given rate of interest and over a given time period is limited, the planning decision is subject to constraint. This means that the opportunity cost of capital will differ from the interest rate. Furthermore complementary and supplementary relationships between projects become important, and combinations of different projects may prove more profitable than individual investments. In particular, the revenues from one project may provide additional funds for investment in some other project at a later stage. This is illustrated by the simple example set out in Table 7.12, given that the farmer's funds are limited to £10 000. Although when considered alone, project X yields the highest gain, a combination of projects Y and Z will yield even more. Thus £10 000 invested in project X would produce a total gain of £5200 at the end of 2 years, assuming constant returns to scale. However, if £10 000 is invested in project Y, £7500 revenue will be earned at the end of year 1 and this can be invested in project Z. The total gain would then be £5000 from project Y plus £1500 from project Z, which is £6500.

2. *Preference for a stable income*

This means that a farmer may choose a particular combination of investment activities simply because the returns are more evenly distributed over time, although a larger total gain could be earned from some other combination. Again referring to Table 7.12, such a farmer might choose to invest in project Y alone since this yields a regular annual

return of £75 for each £100 invested, whereas other combinations of projects yield less regular patterns of returns. Clearly, such preferences cannot be reflected by compounding or discounting at a constant rate of interest.

TABLE 7.12
Choice of projects under capital rationing

		Project X	Project Y	Project Z†
Cash flows (£), Year number:	0	−80	−100	−100
	1	−20	+75	+120
	2	+152	+75	
Terminal gain (terminal net worth at zero interest) (£)		52	50	20
Internal rate of return (per cent)		26	32	20

† Note that this project may be delayed for a year if desired.

3. *Indivisibilities and mutual exclusiveness*

In the absence of capital rationing, indivisible and mutually exclusive projects may be compared in terms of their net present values or incremental internal rates of return. However, when the supply of investible funds is limited, it is by no means certain that these measures will lead to the most profitable choice. Thus the existence of indivisibilities, and projects which are mutually exclusive, further constrains the planning problem. For instance, if we assume that the projects, for which the cash flows are shown in Table 7.12, are indivisible, and the farmer has only £180 available for investment, the maximum gain is obtained by combining project X and project Y, the £20 cost of project X at the end of year 1 being derived from project Y. However, this solution is not immediately obvious and clearly would be less so in a more realistic and therefore more complicated problem.

4. *Risk*

The lack of perfect knowledge about future outcomes has already been mentioned as a problem in planning capital investment.

The first two of these four problems can be accommodated within the linear programming model, although since capital investment is involved it will be a multi-stage linear programme. Thus the simple problem set out in Table 7.12 can be expressed as a two-period linear programming problem as shown in Table 7.13. We again assume that the total funds available at the outset amount to £10 000 and that the projects are divisible and can be expanded to scale and combined like ordinary linear programming 'activities'.

There are two constraints, one on 'initial funds' and the other on 'funds at the end of year 1' which are available for investment in year 2.

TABLE 7.13

The two-period linear programme

	Con-straint level	Activities				
		Project X	Project Y	Project Z_1	Project Z_2	Transfer of funds
Terminal revenue at end year 2 (£)	—	1.52	0.75	0	1.20	0
Initial funds (£)	10 000	0.8	1	1	0	1
Funds at end year 1 (£)	0	0.2	−0.75	−1.20	1	−1

Project Z can be carried out either in year 1 as project Z_1, or in year 2 as project Z_2, or both. The 'transfer of funds' activity allows for the possibility of saving 'initial funds' for investment in the second year of the plan. Projects Y and Z_1 and the transfer of funds activity all contribute to the supply of funds in the second year so these activities have negative coefficients against the 'funds at end of year 1' constraint.

The objective is to maximize terminal assets, which in this case consist only of cash revenues. By maximizing terminal assets rather than their present values we avoid all problems of choosing an appropriate discount rate. No interest is charged explicitly but each project generates its own rate of return. The opportunity costs of limited funds are determined, as shadow prices, within the planning process.

The optimal solution for this simple example is to invest the £10 000 of initial funds in project Y to yield both £7500 at the end of year 1 and a further £7500 of terminal revenue at the end of year 2. The £7500 revenue at the end of the first year is invested in project Z_2 to yield a further £7500 × 1·2 = £9000 of terminal revenue at the end of year 2, giving a

total of £16 500. Thus the initial investment of £10 000 earns an extra £6500 or 65 per cent return over the 2 years of the plan.

An increase, or a decrease, of £1 in the initial funds will result in a rise, or a fall, in the terminal assets of £1.65. In other words the shadow price for initial funds is £1.65; they earn 65 per cent over the 2 years. However, the shadow price for funds at the end of year 1 is only £1.20 earned by investing in project Z_2. Thus the rate of return on investment is 20 per cent in year 2 and 65 −20 = 45 per cent in year 1. Note that the rate of return in year 1 bears no apparent relationship with the internal rates of return for any of the projects. The opportunity cost for the 'transfer of funds' activity is £0.45 per £1 transferred since this is the income forgone by delaying investment until the end of the first year.

Neither project X nor project Z_1 appears in the optimal solution and the introduction of either of them would reduce terminal assets. For instance, the opportunity cost of investing £1 in project X is

$$(£1.65 \times 0{\cdot}8) + (£1.20 \times 0{\cdot}2) = £1.56,$$

while the terminal revenue is only £1.52, so assets would be reduced by £(1.56 −1·52) = £0.04 for each £1 spent in this way. For project Z_1 the opportunity cost is £1.65 while the extra terminal revenue is £(1.2 × 1.2) = £1.44 so the net loss is £(1.65 −1.44) = £0.21 for each £1 invested.

Much larger problems can be dealt with in this way. The number of alternative investments, and other activities can be increased, other constraints such as land or seasonal labour can be incorporated for each year of the plan duration, and the plan duration can be extended to more than 2 years. The ultimate limit on the size of problem that can be handled is the computer capacity.

Formulated like this, the multiperiod linear programme allows for the productivity of capital but does not incorporate the farmer's pattern of time-preference for consumption. However, this is easily achieved.

The methods are illustrated by the linear programming problem set out in Table 7.14, which is adapted from the previous example by omitting the 'transfer of funds' activity and by adding 6 new activities and 2 constraints. There is no need to include the 'transfer of funds' activity because project Z_1 is, in effect, a more profitable way of transferring funds to the end of the first year. Project Z_1 may be said to 'dominate' the transfer of funds activity. Two consumption activities are shown in Table 7.14, 'basic consumption' and 'luxury consumption'. Both these activities produce notional gains in terminal assets, which may be interpreted as their contribution to the total stock of the farmer's satisfaction. These notional gains will appear in the total value of terminal

TABLE 7.14

The incorporation of consumption, lending, and borrowing activities

	Constraint level	Project X	Project Y	Project Z_1	Project Z_2	Basic consumption	Luxury consumption year 1	Lend year 1	Lend year 2	Borrow year 1	Borrow year 2
Terminal assets at end year 2 (£)	—	1.52	0.75	0	1.20	2.00	1.44	0	1.10	0	−1.30
Initial funds (£)	10 000	0.8	1	1	0	1	1	1	0	−1	0
Funds at end year 1 (£)	0	0.2	−0.75	−1.20	1	1	0	−1.10	1	1.30	−1
Limit on basic consumption (£)	1000	0	0	0	0	1	0	0	0	0	0
Limit on borrowing in year 1 (£)	3000	0	0	0	0	0	0	0	0	1	0

assets for the solution and must be subtracted in order to arrive at the value of tangible assets.

The valuation placed on 'basic consumption', necessary for survival, is likely to be high. From a practical planning point of view, it must be high enough to ensure that the basic consumption activity will appear in the optimal solution. For the present example, the notional terminal value of £2 for each £1 consumed per year is sufficient to force basic consumption into the plan. Consumption needs are assumed to be £1000 per year so there is a constraint limiting the activity to this level. Alternatively basic consumption could be incorporated into the plan simply by subtracting the required amount fom the funds available for investment in each year.

Luxury consumption is shown as a possible alternative to investment only in year 1. A similar activity could easily be incorporated for the second year but it is omitted for simplicity. For this example it is assumed that the farmer's marginal rate of time-preference is 20 per cent. Hence for each £1 consumed over and above his basic requirements his stock of satisfaction is increased by £1.20 at the end of the first year and by £1.20 × 1·2 = £1.44 at the end of the second. In practice this means that if the incremental rate of return on investment in the first year should fall below 20 per cent he would consume rather than invest his funds. The activity could be made subject to a constraint if it were thought likely that a 'saturation' level of luxury consumption might be reached. To allow for a diminishing marginal rate of time-preference with increased annual consumption, a series of luxury consumption activities could be included in each year of the planning period. They would differ in respect of the rate of time-preference (terminal satisfaction per £1 consumed) and all but the last, with the lowest rate of time preference, would be subject to constraints. Thus the luxury consumption activity shown in Table 7.14 could be made subject to a maximum constraint of, say, £2000. Luxury consumption beyond this rate could then be represented by a further activity with a terminal value per £1 of only £1.21 for instance.

The problem set out in Table 7.14 also includes lending and borrowing activities in both years of the planning period. The rate of interest for lending is assumed to be 10 per cent so for each £1 lent there is a return of £1.10 a year later. Conversely, borrowing activities contribute to funds (and hence have negative coefficients) at the stage when credit is obtained but must be repaid with interest, assumed to be 30 per cent in this case, a year later. All these activities may be made subject to constraints and there may be more than one borrowing or lending activity in any year if

interest rates are expected to vary according to the sum involved However for simplicity only one constraint is included in Table 7.14, restricting initial borrowing to £3000.

The optimal solution to the problem set out in Table 7.14 is to borrow £3000 in year 1, and this together with the initial £10 000 gives a total of £13 000. Out of this £1000 is required for basic consumption, leaving £12 000, which is invested in project Y. At the end of the first year the return from project Y is £12 000 × 0·75 = £9000. This sum is used to meet the £1000 basic consumption in the second year and to repay the £3000 loan with 30 per cent interest, making £3900. Hence

$$£(9000 - 1000 - 3900) = £4100$$

is left and this is invested in project Z_2. At the end of the second year the terminal assets amount to

$$(£12\,000 \times 0{\cdot}75) + (£4100 \times 1{\cdot}2) + (£1000 \times 2{\cdot}0) = £15\,920.$$

Note, however, that the value of basic consumption £1000 × 2·0 is purely notional and should be subtracted to arrive at the tangible assets, which in this case amount to £13 920.

As before, projects X and Z_1 do not appear in the optimal plan but the borrowing of funds in year 1 enables a larger amount to be invested in projects Y and Z_2. The only new activity in the optimal plan is basic consumption. The luxury consumption activity does not appear in the optimal plan since the £1.44 notional satisfaction derived per £1 consumed is less than the opportunity cost of £1.65. In other words the marginal rate of time preference of 20 per cent is less than the rate of return which can be obtained by investing in project Y. Likewise the rate of interest on lending outside the farm, of 10 per cent, is less than the rate of return that can be earned by investing within the farm business in either year. It does not pay to borrow in year 2 at 30 per cent to invest in project Z_2 and earn only 20 per cent return. However each £1 borrowed in year 1 earns £1.65 for an opportunity cost of £1.30 × 1·20 = £1.56 and hence yields a net gain of £0.09, or 9 per cent over 2 years. Note that without a constraint on this borrowing activity there would be no limit to the scale of the business and hence no finite, optimal plan.

In this example we have avoided the problem of valuing physical assets by assuming that all projects are completed within the 2-year duration of the plan. No further revenues are expected after the end of the planning period. However this need not be the case. Had we allowed the possibility of investing in project X or project Y at the end of the first year, these projects would not be completed at the end of the second year; revenues

would still be expected. Hence these projects would have residual values as productive assets at the end of year 2. The problem then arises of estimating these residual values. In theory the value of a physical asset at any point in time is the discounted value of expected subsequent returns. But this means that in order to estimate residual values it is necessary to predict the appropriate discount rate or shadow price of funds, after the terminal point of the plan.

Thus we are back with the problem of circularity. The choice of projects depends upon their terminal values and therefore upon the shadow prices of funds *but* the shadow prices of funds depend upon the choice of projects. In practice it means that some arbitrary estimate of

TABLE 7.15

Mutually exclusive and complementary projects

	Constraint level	Activities		
		Project A	Project B	Project C
Mutually exclusive	$1 \geqq$	1	1	0
Complementary	$0 \geqq$	−1	0	+1

discount rates or of terminal values must be used. However, it would appear that the longer the duration of the plan the smaller will be the effect of residual values. Furthermore, as we have seen, the problem can be avoided completely by assuming that all terminal assets are held in the form of cash.

When projects are indivisible, *integer programming* or mixed linear/integer programming may be used rather than ordinary linear programming. It is then possible to specify that certain projects can be varied only by integer (whole-number) amounts. However, the incorporation of integer variables considerably slows down the process of reaching an optimal solution, and hence increases the cost. In some cases ordinary linear programming provides a reasonable approximation and integer programming is not justified. For instance in an optimal solution the error involved in rounding say 132·46 cows to 132 cows may reasonably be ignored. On the other hand, serious errors might be involved in rounding say 2·46 men employed to 2 or 3.

The importance of integer variables is not, however, limited to such cases. Of perhaps more importance is the possibility of allowing for

incompatibility or dependency between projects. The methods are illustrated by the numbers given in Table 7.15. Each of the three projects A, B, and C is assumed to be indivisible and the use of integer programming ensures that fractions of a unit cannot be included in the solution. Projects A and B are assumed to be incompatible or mutually exclusive, and the first row of the table ensures that one or other but not both projects can appear in the final plan. Since it is impossible for negative quantities or fractions of a unit of either project to occur, there are only 3 possible ways in which this constraint can be met, namely 1 unit of project A and none of B, or 1 unit of project B and none of A, or neither A nor B.

Projects A and C are assumed to be dependent or complementary in that it is undesirable or impossible to invest in project C without first establishing project A. The second row of Table 7.15 makes it impossible for project C to enter the solution until project A is included. Thus integer programming does have certain advantages over ordinary multiperiod linear programming in solving investment problems, but it also has disadvantages in terms of time and cost.

As an alternative to multiperiod linear or integer programming, the technique of *dynamic programming* may be used in planning investments. This too is a method of finding an optimal plan, in this case one which maximizes terminal net worth, subject to constraints. However, dynamic programming differs from the other methods we have discussed in that the problem is treated as a sequence of optimizing decisions. For very simple problems the technique may be illustrated by means of a decision tree. Figure 7.9 represents the decision tree for the data set out in Table 7.12, given that each project is indivisible and only £180 is available for investment. Again the time and cost involved in solving a real planning problem using dynamic programming may be substantial. However, dynamic programming has the advantage that it can more readily take variability of outcomes into account than can linear and integer programming. For instance a simple probabalistic decision tree was used in Fig. 4.5 (p. 147).

However, as remarked in Chapter 4, most real-life whole-farm planning problems are so complex, particularly when risk is taken into account, that even with the aid of powerful computers they lie beyond the reach of maximizing techniques unless drastic simplifying assumptions are made. Furthermore, farmers may be concerned with 'satisficing' rather than optimizing. Nevertheless they will be interested in predicting the outcomes of alternative plans which may have been chosen on some fairly informal and subjective basis. In view of the complexity of the

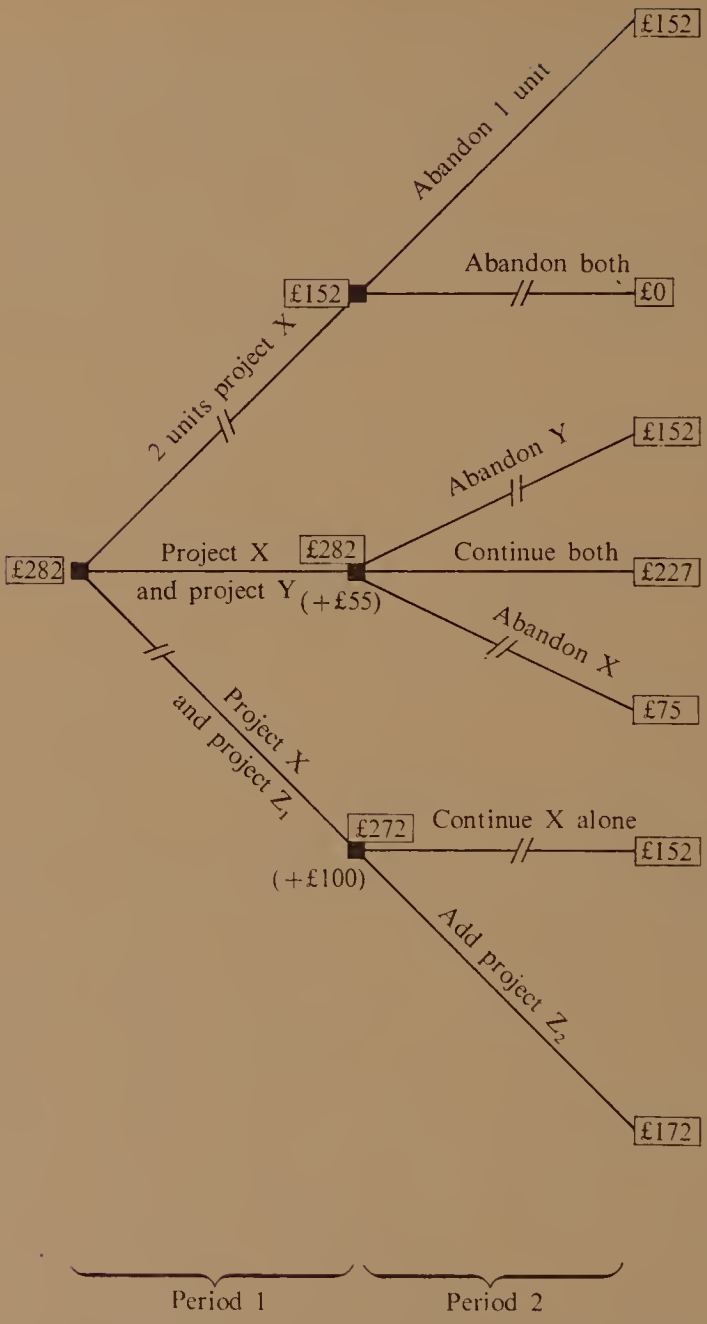


FIG. 7.9. Decision tree for investment. Although no probabilities are involved, a simple sequential planning problem (like that given in Table 7.12 in its integer form) may be solved using a decision tree. More complex problems may be solved by 'dynamic programming' which is based on the same principles of backward induction and 'pruning' of the less profitable branches.

planning situation, mere prediction may involve a great deal of calculation and the technique of *computer simulation* may be used for this purpose. The great advantage of a simulation model over most existing optimization models is that it can be made to correspond more closely with the real-world system being studied. However this introduces great size and complexity into the model and necessitates the identification and quantification of all the factors involved.

Summary

1. The introduction of capital (time) adds many new dimensions to the analysis of production decisions.
2. For some purposes the problems of multi-dimensionality may be overcome by using discounting or compounding techniques to reduce an entire income stream to a single number (net present value or terminal net worth) given that the cost of capital (interest rate) is known.
3. For a single project the optimal time for replacement is when the extra (marginal) gain from prolonging the project falls to equality with the average annual return that can be earned with the funds and other resources involved (opportunity cost).
4. The multi-dimensional character of capital raises severe problems of measurement of the quantity involved and the rate of return.
5. The stock of capital may be increased by investment but the amount of savings and credit available for investment over a given time period will be limited, and subject to increasing marginal cost.
6. Various operational research techniques have been proposed for whole-farm planning over time, one of which is multiperiod linear programming.

Mathematical appendix

Basic notions

1. Let

K_t = stock of capital of which $x \,.\, K_t$ is fixed and $(1 - x) K_t$ is working capital at end of year t;

M_t = margin of receipts over current costs in year t;

D_t = depreciation in year t.

n.b. we assume all these can be measured in the same units. If h = annual depreciation rate,

$$\text{then } D_t = h \,.\, x \,.\, K_{t-1},$$

since capital stock at start of year t is that held at the end of the previous year, $t - 1$. The net return in year t, R_t is given by

$$R_t = M_t - D_t.$$

This is usually expressed as a rate of return r_t thus:

$$r_t = R_t/K_{t-1}. \tag{7.1}$$

Alternatively if i = rate of interest = opportunity cost of capital, then profit

$$\pi_t = R_t - i \,.\, K_{t-1}. \tag{7.2}$$

In static conditions all these quantities are constants or stable expected values, so the subscript t is not strictly necessary.

2. Now if

C_t = consumption in year t;

I_t = investment in year t,

then

$$M_t = I_t + C_t.$$

If there is no borrowing,

$$\frac{\mathrm{d}K_t}{\mathrm{d}t} = K_t - K_{t-1} = I_t - D_t.$$

With borrowing (B_t in year t) we have,

$$\frac{\mathrm{d}K_t}{\mathrm{d}t} = I_t - D_t - B_{t-1} + B_t. \tag{7.3}$$

Compounding and discounting

Let

P = original loan (principal) in £ or other suitable units;

i = annual interest rate expressed as a decimal fraction, (e.g. 10 per cent = 0·10)[13].

[13] This assumes that interest is paid annually; time is measured in discrete, yearly steps. Alternatively interest may be compounded continuously, using the formula:

$$B_t = P\,\mathrm{e}^{it},$$

where P = original loan, t = time measured continuously, i = continuous rate of interest, B_t = cumulated debt at time t and e = base of natural logarithms—approximately 2·718. Rearrangement yields the formula for continuous discounting, namely:

$$P = B_t\,\mathrm{e}^{-it}.$$

Then the total debt after 1 year B_1 is the principal plus interest:

$$B_1 = P + i \,.\, P = P(1 + i).$$

If no repayments are made, the debt after 2 years, B_2 will be

$$B_2 = B_1(1 + i) = P(1 + i)^2.$$

Similarly the debt after 3 years B_3 is calculated as

$$B_3 = B_2(1 + i) = P(1 + i)^3.$$

From this it should be apparent that the debt at the end of any number t of years, may be calculated by the general formula for compounding, given that i remains constant:

$$B_t = P(1 + i)^t. \tag{7.4}$$

This formula was used to derive Table A.1 (p. 342).

The above formula (eqn (7.4)) may be rearranged to allow calculation of the present value P of a given sum B_t expected after t years. Thus dividing both sides of eqn (7.4) by $(1 + i)^t$ gives

$$P = \frac{B_t}{(1 + i)^t}. \tag{7.5}$$

This is the general formula for discounting, used to derive Table A.2 (p. 343).

Cash flows and annuities

1. Now consider a series of annual cash flows over t years, M_0, M_1, M_2, M_3, . . . M_t starting with M_0, the initial investment, and finishing with M_t, which includes the salvage value. These cash flows may be positive (receipts) or negative (costs). In the simplest case we assume that receipts earn the same rate of interest i as is charged on costs, and that i is constant over time.

At the end of t years the initial investment M_0 will amount to $M_0(1 + i)^t$ (substituting M_0 for P in eqn (7.4)). However, M_1 is compounded only over $t - 1$ years to give $M_1(1 + i)^{t-1}$. Subsequent cash flows are compounded over correspondingly shorter periods. The total terminal net worth (TNW) is the sum of all these quantities, thus:

$$(TNW) = M_0(1 + i)^t + M_1(1 + i)^{t-1} + \\ + M_2(1 + i)^{t-2} + \ldots + M_t. \tag{7.6}$$

2. The net present value (NPV) may be derived using discounting in a similar manner. Thus, dividing both sides of eqn (7.6) by $(1 + i)^t$ gives

$$(NPV) = \frac{(TNW)}{(1+i)^t} = M_0 + \frac{M_1}{(1+i)} + \frac{M_2}{(1+i)^2} + \ldots + \frac{M_t}{(1+i)^t}. \quad (7.7)$$

3. In the special case of an annuity all cash flows (omitting the initial investment) are equal:

$$A = M_1 = M_2 = \ldots = M_t.$$

Hence by substitution in eqn (7.6) we obtain the terminal net worth of an annuity (TA) as:

$$\begin{aligned}(TA) &= A(1+i)^{t-1} + A(1+i)^{t-2} + \ldots + A(1+i) + A \\ &= A\{(1+i)^{t-1} + (1+i)^{t-2} + \ldots + (1+i) + 1\}.\end{aligned}$$

But this equation may be simplified by multiplying both sides by $(1 + i)$, thus

$$(TA)(1+i) = A\{(1+i)^t + (1+i)^{t-1} + \ldots + (1+i)^2 + (1+i)\}$$

and then subtracting the original equation,

$$\text{subtracting } (TA) \quad = A\{ \quad (1+i)^{t-1} + \ldots + (1+i) + 1\}$$

gives

$$(TA)(1 + i - 1) = A\{(1+i)^t - 1\}.$$

But

$$(TA)(1 + i - 1) = (TA) \,.\, i$$

therefore

$$(TA) = \frac{A\{(1+i)^t - 1\}}{i}. \quad (7.8)$$

Table A.3 (p. 344) is derived from this formula.

4. We may divide both sides of eqn (7.8) by $(1 + i)^t$ to give the present value of an annuity (PA):

$$(PA) = \frac{(TA)}{(1+i)^t} = \frac{A\{(1+i)^t - 1\}}{i(1+i)^t}. \quad (7.9)$$

Table A.4 (p. 345) is derived from this formula.

Rearrangement (i.e. multiplying both sides by

$$\frac{i(1+i)^t}{(1+i)^t - 1}$$

and substituting (NPV) for (PA)) yields a formula for converting a given (NPV) into an equivalent annuity:

$$A = \frac{(NPV) \cdot i(1+i)^t}{(1+i)^t - 1}. \tag{7.10}$$

Perpetuities

1. If an annuity is expected to continue into the indefinite future, as is the case with land rents or any other continuous stream of income net of depreciation, it is known as a perpetuity. The present value of a perpetuity $(PA)_\infty$ is derived as follows.

Eqn (7.9) may be rewritten (dividing the right-hand side through by $(1+i)^t$) as

$$(PA) = \frac{A}{i}\left\{1 - \frac{1}{(1+i)^t}\right\}.$$

But

$$i > 0, \quad \text{so} \quad \frac{1}{(1+i)} < 1,$$

and therefore as

$$t \to \infty, \quad \frac{1}{(1+i)^t} \to 0.$$

Thus at the limit

$$(PA)_\infty = \frac{A}{i}. \tag{7.11}$$

2. *Alternative proof:* The terminal net worth (TNW) of an asset which yields an annuity A per year for t years and a salvage value S_t at the end is given by

$$(TNW) = \frac{A\{(1+i)^t - 1\}}{i} + S_t$$

(from eqn (7.8)). Hence

$$(TNW) - S_t = \frac{A\{(1+i)^t - 1\}}{i}. \tag{7.12}$$

But for a perpetuity the asset value will be the same after t years as it was at the start. Therefore

$$(PA)_\infty = \frac{(TNW)}{(1+i)^t} = S_t.$$

Substitution in eqn (7.12) gives

$$(PA)_\infty\{(1+i)^t - 1\} = \frac{A}{i}\{(1+i)^t - 1\}$$

and therefore

$$(PA)_\infty = \frac{A}{i}.$$

Note that this means that $i = A/(PA)_\infty$. For static conditions this equation is comparable with eqn (7.1).[14]

The internal rate of return

$(IRR) = r$ is defined as the rate of interest at which (TNW) (eqn (7.6)) and (NPV)(eqn (7.7)) are equal to zero; i.e. from eqn (7.6):

$$0 = M_0(1+r)^t + M_1(1+r)^{t-1} + M_2(1+r)^{t-2} + \ldots + M_t.$$

This is a polynomial equation in $(1 + r)$ of degree t. It therefore has t roots, or possible solutions for r. For instance, where $t = 2$ we have a quadratic equation which has two roots, or solutions: let $M_0 = -100$; $M_1 = +225$; $M_2 = -126$ as in Table 7.6, then

$$0 = -100(1+r)^2 + 225(1+r) - 126$$

so

$$0 = \{-10(1+r) + 10{\cdot}5\}\{10(1+r) - 12\}.$$

Hence either

$$(1+r) = 1{\cdot}05 \quad \text{and} \quad r = 0{\cdot}05$$

[14] With continuous compounding or discounting we have, from time zero to time T:

$$(TNW)_T = \int_0^T M(t)\,e^{it}\,.\,dt$$

and

$$(NPV)_T = \int_0^T M(t)\,e^{-it}\,.\,dt.$$

For an annuity this becomes

$$(PA)_T = \int_0^T A\,e^{-it}\,.\,dt = \frac{A(1 - e^{-iT})}{i}$$

and, for a perpetuity, since $0 < i < 1$, as $T \to \infty$, $e^{-iT} \to 0$ so

$$(PA)_\infty = A/i$$

or

$$(1 + r) = 1{\cdot}20 \quad \text{and} \quad r = 0{\cdot}20.$$

However, our theory requires that $r > 0$ (r is positive) and this clearly requires that $(1 + r)$ is positive, so we may ignore any non-positive roots. Now, when there is only one change of sign in the series of cash flows, from negative to positive or vice versa, there can only be one positive root and hence there is a unique positive value for r. Multiple solutions as in the above example can only occur where there is more than one change of sign.[15] These multiple solutions give rise to the problem of double switching.

Optimization over time

1. Consider a single, discrete project with cash flows, or margins, $M_0, M_1, \ldots, M_m, \ldots, M_T$ assuming that:

(a) these margins (adjusted for changes in salvage value—see below) diminish from maturity, at year m, onwards, where $m \leqq T$;
(b) beyond year T, margins (adjusted for changes in salvage value) are negative so it cannot pay to continue production after year T.

Note that if the objective were to maximize total profit from a single, unrepeated run of the project (whether discounted or not) the optimum project life would be T years.

2. However, if the project may be repeated and the objective is to maximize the total discounted profit Z_0 from a continuous series of runs of the project, the optimum project life may be less than T years. Let

i = opportunity cost of capital = constant;

S_t = salvage value in year t (not included in the margin M_t);

$(NPV)_t$ = net present value given a single project life of t years.

Then

$$(NPV)_t = M_0 + \frac{M_1}{(1 + i)} + \frac{M_2}{(1 + i)^2} + \ldots + \frac{M_t + S_t}{(1 + i)^t}$$

(as eqn (7.7)).

[15] Descartes's change of sign rule states that there will be as many positive real roots as there are changes of sign, when all terms with zero coefficients have been supressed. Note however that where $(1 + r) < 1$, $r < 0$, so we may also ignore those positive roots which are less than unity.

However, eqn (7.10) may be used to convert this into an equivalent annuity over t years A_t. Since the project may be repeated after t years A_t is, in effect, a perpetuity, so

$$Z_0 = \frac{A_t}{i}. \tag{7.13}$$

Since i is assumed to be constant Z_0 is maximized by maximizing A_t.

3. As in the section on basic notions let

$$\pi_t = R_t - i \,.\, K_{t-1} = M_t - D_t - i \,.\, K_{t-1}$$

(from eqn (7.2)), where π_t = profit in year t, D_t = depreciation in year t, and K_{t-1} = capital at end of year $t-1$ or start of year t. But in the present context $K_{t-1} = S_{t-1}$ and $D_t = S_{t-1} - S_t$. Hence

$$\begin{aligned}\pi_t &= M_t - (S_{t-1} - S_t) - i \,.\, S_{t-1} \\ &= M_t + S_t - (1+i)S_{t-1},\end{aligned}$$

i.e.

π_t = cash flow adjusted for changes in salvage value.

Now, A^*, the maximum value of A_t, is obtained at optimum project life t^*, where

$$m \leqq t^* \leqq T \quad \text{and} \quad \pi_t = M_t + S_t - \\ - (1+i)S_{t-1} \leqq A^* \text{ for all } t > t^*. \tag{7.14}$$

Notes.

(a) It is really π_t and not M_t which must be diminishing between years m and T for an optimum to exist.

(b) The inequality (7.14) provides a rule which is completely general, e.g. for the point-input, point-output case, referred to on pp. 267 and 285,

$$A^* = M_t = 0$$

and the rule for the optimum project life becomes

$$S_t \leqq (1+i)S_{t-1}.$$

(c) If a new alternative project becomes feasible, with maximum equivalent annuity $A(N)^*$, where $A(N)^* > A^*$, then π_t is compared with $A(N)^*$ to determine when to replace the existing project with the new one.

Multiperiod planning

1. The above approach to optimization over time requires that:

(a) the annual cash flows, M_0 to M_t, reflect the opportunity costs of resource inputs and are predetermined;

(b) the opportunity cost of capital i is constant and predetermined.

Where there are resource constraints, these conditions will not be satisfied; opportunity costs are then dependent on the planning choices made. In particular if there is a limit on borrowing, the opportunity cost of capital may exceed the market rate of interest and may vary from year to year. It is not possible to determine the present value of a capital investment until the opportunity costs of capital over the life of the investment are known, but this requires that the planning problem has already been solved. This is the circular argument involved in evaluating capital.

Whole-farm planning over time requires the *growth path* to be considered.

2. Consider a simple case with a 4-year planning horizon and just 2 projects, A and B, with unit cash flows $M(\mathrm{A})_0$, $M(\mathrm{A})_1$, and $M(\mathrm{A})_2$, and $M(\mathrm{B})_0$, $M(\mathrm{B})_1$, $M(\mathrm{B})_2$, and $M(\mathrm{B})_3$ respectively. The number of units a_t and b_t of each project established in the tth year can be varied continuously and there are constant returns to scale. We assume no borrowing or lending and no resource constraints other than on initial capital K. There is no hoarding so the initial capital and all revenues are either invested or consumed. Let I_t = investment in year t and C_t = consumption in year t, then

$$K = C_1 + I_1. \tag{7.15}$$

But

$$I_1 = -\{a_1 \,.\, M(\mathrm{A})_0 + b_1 \,.\, M(\mathrm{B})_0\}$$

since projects A and B are the only two avenues of investment open. (Note that an investment is a negative cash flow so a minus sign is needed.) Thus we may rewrite eqn (7.15) as

$$K = C_1 - \{a_1 \,.\, M(\mathrm{A})_0 + b_1 \,.\, M(\mathrm{B})_0\} \tag{7.16}$$

Now, the sum available at the end of year 1, Y_1 will be the revenues from projects A and B in year 1 (if any), thus:

$$Y_1 = a_1 \,.\, M(\mathrm{A})_1 + b_1 \,.\, M(\mathrm{B})_1.$$

This sum may also be invested or consumed in year 2:

$$Y_1 = C_2 - \{a_2 \,.\, M(\mathrm{A})_0 + b_2 \,.\, M(\mathrm{B})_0\}. \tag{7.17}$$

Likewise revenues from projects A and B in years 2 and 3 may be invested or consumed:

$$Y_2 = C_3 - \{a_3 \,.\, M(\mathrm{A})_0 + b_3 \,.\, M(\mathrm{B})_0\}; \tag{7.18}$$

$$Y_3 = C_4 - \{a_4 \,.\, M(\mathrm{A})_0 + b_4 \,.\, M(\mathrm{B})_0\}. \tag{7.19}$$

By the end of year 4, revenues will be earned from all investments which have not yet reached the ends of their lives (i.e. since project A has a 2-year life and B a 3-year life; at year 4, there will be no remaining cash flows from investments a_1, a_2, and b_1. Thus:

$$Y_4 = a_3 \,.\, M(\mathrm{A})_2 + a_4 \,.\, M(\mathrm{A})_1 + \\ + b_2 \,.\, M(\mathrm{B})_3 + b_3 \,.\, M(\mathrm{B})_2 + b_4 \,.\, M(\mathrm{B})_1.$$

Apart from these revenues, terminal assets W will also include the terminal values of incompleted projects, $V(\mathrm{A})$ and $V(\mathrm{B})$. So:

$$W = Y_4 + V(\mathrm{A}) + V(\mathrm{B}), \tag{7.20}$$

where

$$V(\mathrm{A}) = a_4 \,.\, \frac{M(\mathrm{A})_2}{(1+i)} \tag{7.21}$$

and

$$V(\mathrm{B}) = b_3 \,.\, \frac{M(\mathrm{B})_3}{(1+i)} + b_4 \,.\, \left\{\frac{M(\mathrm{B})_3}{(1+i)^2} + \frac{M(\mathrm{B})_2}{(1+i)}\right\}. \tag{7.22}$$

Thus even in this simple example with 2 investment activities (projects) and a 4-year planning horizon there are 12 variables to be evaluated in choosing a growth path: i.e. a_1, b_1, C_1; a_2, b_2, C_2; a_3, b_3, C_3; a_4, b_4, C_4. Moreover, these variables are not independent, since:

(a) only 2 out of the 3 (a_t, b_t, and C_t) can be chosen independently in any one year since the total expenditure is limited by the funds available;

(b) the choices made in any year are constrained by choices made in previous years; Y_t is a function of a_{t-1}, b_{t-1}, a_{t-2}, ...

3. Most dynamic models assume one of two alternative criteria:

(a) *Continuous optimality* with the objective of maximizing the time path of consumption only (e.g. maximize

$$Z = f(C_1, C_2, C_3, C_4))$$

subject to constraints on the terminal stocks e.g. $W \geqslant K$. By making the number of time periods within the planning horizon

large enough the effect of terminal stocks may be almost erased.

(b) *Terminal optimality* with the objective of maximizing terminal assets only (e.g. maximize

$$W = Y_4 + V(\mathrm{A}) + V(\mathrm{B})),$$

subject to constraints on the pattern of consumption (e.g. $C_t =$ constant and fixed). n.b. The problem of capital valuation using an arbitrary estimate of i, the opportunity cost of capital, still arises in estimating $V(\mathrm{A})$ and $V(\mathrm{B})$.

The two criteria may be combined if the producer's personal rate of time preference ρ is known, (for simplicity assume ρ is constant). Then terminal 'satisfaction' at the end of 4 years

$$= C_1(1+\rho)^3 + C_2(1+\rho)^2 + C_3(1+\rho) + C_4.$$

Thus total terminal 'wealth', which is now the quantity to be maximized is a composite comprising (a) terminal income = cash, (b) terminal physical assets, and (c) terminal 'satisfaction'. For our simple example.

$$(TW) = Y_4 + V(\mathrm{A}) + V(\mathrm{B}) + C_1(1+\rho)^3 + \\ + C_2(1+\rho)^2 + C_3(1+\rho) + C_4.$$

4. *Notes.*

(a) Borrowing and lending activities are readily incorporated: a separate borrowing (or lending) activity for each year of the plan; thus borrowing in any year involves two cash flows: $M_0 = +1$ and $M_1 = -(1+i)$. For lending the cash flows are $M_0 = -1$ and $M_1 = (1+i)$. n.b. i for borrowing may differ from i for lending.

(b) Given the assumption of constant returns to scale, if (TW) can be increased by borrowing (in effect if internal rate of return, $r > i$) and there is no limit on the amount that can be borrowed, then clearly (TW) can be increased indefinitely: there is no optimum unless there are limitations of the type noted in (c) below.

However, if borrowing is limited to a fixed proportion $g =$ gearing ratio of owned assets for risk avoidance reasons, then a limit is imposed.

(c) Limitations on the amounts of other resources, e.g. land, labour, or management, may also limit growth. They may be treated as constraints as in Chapter 2.

Suggestions for further reading

BIERMAN, H. and SCHMIDT, S. (1975). *The capital budgeting decision* (4th ed.). Macmillan, New York.

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HARCOURT, G. C. and LAING, N. F. (1971). *Capital and growth.* Penguin Books, Harmondsworth.

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ROBINSON, J. (1956). *The accumulation of capital.* Macmillan, London.

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8. Management

The managerial function

This chapter is concerned with the role of management in the production process, the meaning of profit, and the determinants of farm size and growth. It should be remarked at the outset that there is no generally accepted, unified theory regarding these matters; rather there are many alternative, sometimes conflicting, hypotheses. The ideas presented in this chapter are therefore only tentative.

Physical resources of land, labour, and capital are not productive unless they are organized and co-ordinated by someone who makes the necessary decisions and sees that they are carried out. On a small private farm, typical of a large part of the world's agriculture, these services of organization and co-ordination are provided by a single individual, namely the farmer. This may be contrasted with the manufacturing industry, for instance, where a typical firm is controlled by a whole hierarchy of decision makers. In such a firm some of these decision makers are hired managers who are paid a regular salary, while others, the owners of the firm, receive an uncertain, residual income. They are known as 'entrepreneurs'. Although the farmer embodies both kinds of decision maker, it is nonetheless useful for some purposes to separate the functions of management and entrepreneurship.

The function of management is essentially that of administration of the business, day-to-day decision making, organization, and supervision of the labour force and other physical inputs. The important characteristic which distinguishes management from entrepreneurship is that managers may be hired. This means that although the farmer does not hire a manager but performs the managerial functions himself nonetheless he could, in theory, hire himself out as a manager to someone else. Hence his managerial input has an opportunity cost, namely the amount he could earn working as a manager for someone else.

Many economists would regard management, defined in this way, as simply a part of the total labour input. Indeed this assumption was implicit in the discussions of population growth and of the size of the agricultural sector in Chapter 6 on labour. However, when we consider the individual farm, it is not really meaningful to measure management

inputs in terms of man-hours. The achievements of a manager depend upon his ability rather than the number of hours he works. This managerial ability varies from one person to another so some means of assessment must be found if the quantity of resources each manager controls is to be related to his ability. In a community of small private farms the responsibility for this assessment of ability falls upon the land-tenure and rural credit institutions responsible for the allocation of land and capital. In larger organizations such as state farms the responsibility for assessing a manager's ability falls upon his superiors in the hierarchy. Generally speaking, the assessment can only be based on past performance. Unfortunately there is no guarantee that a man who is

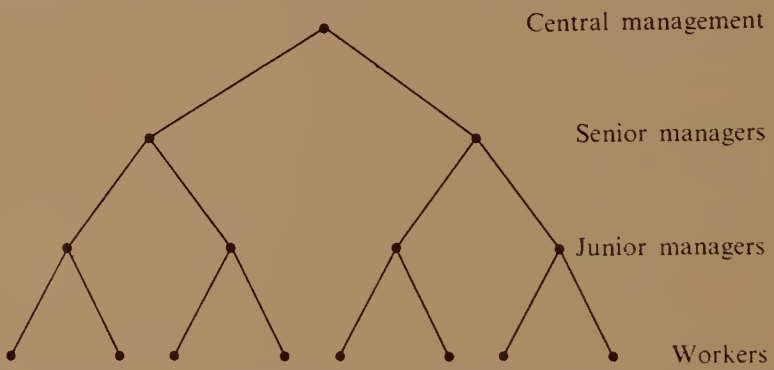


FIG. 8.1. A managerial pyramid. As a business grows so too must the number of managers but the pyramid grows both in width and in height (i.e. the number at each rank and the number of ranks).

successful in managing a productive unit of one size will be equally successful in managing a unit of different size.

In fact the services provided by management vary from one scale of operation to another. Thus a small farmer with no hired labour merely has to organize himself and the necessary materials and equipment. With a larger labour force, the farmer must motivate and supervise the others. As the size of the labour force grows the tasks of communication and co-ordination grow more rapidly. Eventually a whole hierarchy of managers is needed, with long chains of command and communication between the senior manager and the workers in the field (see Fig. 8.1). Thus the ratio of managers to workers is increased.

Entrepreneurship and profit

The functions of entrepreneurship are those of determining over all policy objectives, seeking out new opportunities for gain and bearing the associated risks. These services, though still needed, are not provided in

the same way in a fully centrally planned society but they are essential to the operation of a free market system. Entrepreneurship is different from other resources in that it cannot be hired. Indeed, there is no way in which entrepreneurial capacity can be transferred from one person to another. An entrepreneur is an individual who contracts to pay incomes to others in return for specified services and expects to earn a surplus over these contractual costs which will reward him for his own resource inputs. His own resource inputs may consist of land, labour, capital, or management or any combination of these. However modern theorists completely reject the old notion that ownership of capital is the essence of entrepreneurship. After all a man may establish himself as a farmer or any other kind of entrepreneur by borrowing all the necessary capital. The essential feature is that he is a residual, or non-contractual income recipient, which means that he bears the main risks involved in production.

This is not to imply that contractual claims are always honoured or enforced. However, the risks that contracts may be neither honoured nor enforceable are generally allowed for by including a risk premium in the agreed price. The entrepreneur's reward for risk bearing, on the other hand, is included in his non-contractual income, which elsewhere in this book we have called 'profit'. Now it may be useful to look a little more closely at what profit is. In fact four elements may be distinguished, in theory if not in practice. First there is the implicit return to, or opportunity cost of, resources of land, labour, capital, and management, which are owned by the farmer himself. This is the rate of earnings he would obtain if he gave up his farm and hired his resources out to other entrepreneurs. Secondly there are gains or losses made on past, irreversible capital investments and remaining after due allowance has been made for the costs of interest and depreciation. This element of profit was discussed in the previous chapter. Thirdly, there is the reward for entrepreneurship including innovation and risk bearing, and fourthly there is a surplus or deficit which varies from year to year reflecting temporary, chance variations in contractual costs and revenues. When decisions are made so as to maximize expected profit, only the first three of these elements can be taken into account, the last being outside the farmer's control and only appearing after the decision has been made and the action taken.

We may note that the implicit returns to the farmer's own resources (the first element) are costs which must be met regularly and continuously in order to justify keeping the resources in their present use, whereas the gains or losses on past capital investments (the second element) are in a

sense only temporary. Eventually the existing stock of capital will be used up or worn out and discarded, at which stage any unpremeditated gains or losses associated with that stock will cease. New investments will be made in the light of new, changed prices and technology. However in a situation of continuous change and continuous new investment, an equilibrium situation is never attained. Profits or losses on past investment may be a permanent feature of the production process.

In a world of perfect knowledge and prediction there would be no scope for the exercise of entrepreneurial talent so no reward could be expected: the third element of profit would not exist.[1] Nevertheless managerial inputs would still be needed to administer the production process. In the real world of change and uncertainty entrepreneurship is a necessary input which may or may not receive a positive financial (or physical product) reward. Most probably some entrepreneurs, particularly the more able, will generally earn positive rewards, while others may make losses.

Individuals will vary in the *expected* rate of reward they require to persuade them to remain self-employed; in other words the supply price for entrepreneurship varies from one individual to another.[2] For instance one who is risk averse and would really prefer a stable secure income is likely to have a relatively high supply price. Alternatively one who enjoys the freedom of being his own boss may be willing to accept a negative reward. It should be noted, however, that if his gross revenue consistently falls below his contractual costs he will not be able to continue in business indefinitely. Actual profits earned are not necessarily related directly with supply price so some entrepreneurs will be earning much more than is needed to persuade them to stay in business; they are earning pure economic rents (see Chapter 5). For others, the slightest fall in profits might be sufficient to drive them out of business and into paid employment.

On average it seems likely that the supply price for entrepreneurship will be positive since most people are somewhat risk averse and it is not possible to remain in business if continuous losses are made. However, by definition there is no such thing as a market price for entrepreneurship. Rather there is a somewhat indeterminate relationship between the rate

[1] Note that we are assuming that no individual farmer can prevent access of competitors to a particular product market or a particular resource supply and thereby influence prices to earn himself 'monopoly profit'.

[2] In this context the word 'expected' means 'expected by the individual concerned'. It may or may not coincide with the formal definition of 'expected value' given in Chapter 4.

of profits and the number of entrepreneurs in a particular productive sector, say agriculture. Thus the average rate of profits in agriculture cannot rise or fall much without general attention soon being attracted to the change. This is likely to influence the number of persons desiring to set up in business as farmers. Note that there may be both a transfer of entrepreneurs between agriculture and other sectors and a change in the numbers seeking self-employment.

Managerial ability and farm size

It has been argued by some economists that managerial ability should not be treated as a factor of production. In technical crop and livestock production-response analysis it is assumed that, aside from the random elements generated by variations in *uncontrolled* factors of production, the relationship between physical inputs and physical outputs is stable; 'exactly the same amounts and combinations of factors of production used under exactly the same physical conditions are expected to produce the same physical product.'[3] According to this view it is not possible for nature to produce one result at one time and another result at another time under otherwise identical conditions simply because of differences in some 'ghost' input called management.

The weakness of this argument is that the factors which are 'uncontrolled' in experiments are, at least in part, under the control of the farmer or farm manager. A general production function incorporating every conceivable variable that might come under the farmer's control might well involve too many variables to be determined by an empirical statistical analysis. Thus a wide range of decisions are left to the farmer's judgement. Differences among individuals in their ability to make these decisions will be reflected by *different* outputs from similar input combinations. Incidentally, differences in output achieved for a given level of inputs are sometimes called differences in technical efficiency to distinguish them from variations in the choice of resource combinations which reflect economic or allocative efficiency. The farmer who uses the least-cost combination of resource inputs is said to be economically efficient. To simplify the present discussion we will assume that all farmers are economically efficient so that we may concentrate on differences in technical efficiency.

The following analysis could be applied to an individual enterprise or production unit within the farm or to the whole farm, again assuming

[3] See Johnson, G. L. (1964). A note on non-conventional inputs and conventional production functions. In *Agriculture in economic development* (ed. C. Eicher and L. Witt). McGraw-Hill, New York.

that some satisfactory method is found for aggregating different products into a single measure of output. By assuming that all farmers are economically efficient in using the least-cost combination of inputs, we may measure all inputs along a single scale, as shown in Fig. 8.2. The technically optimum production function is given by the line OS. From a purely technical point of view it seems reasonable to assume constant returns to scale. Hence this technical efficiency frontier is shown as a

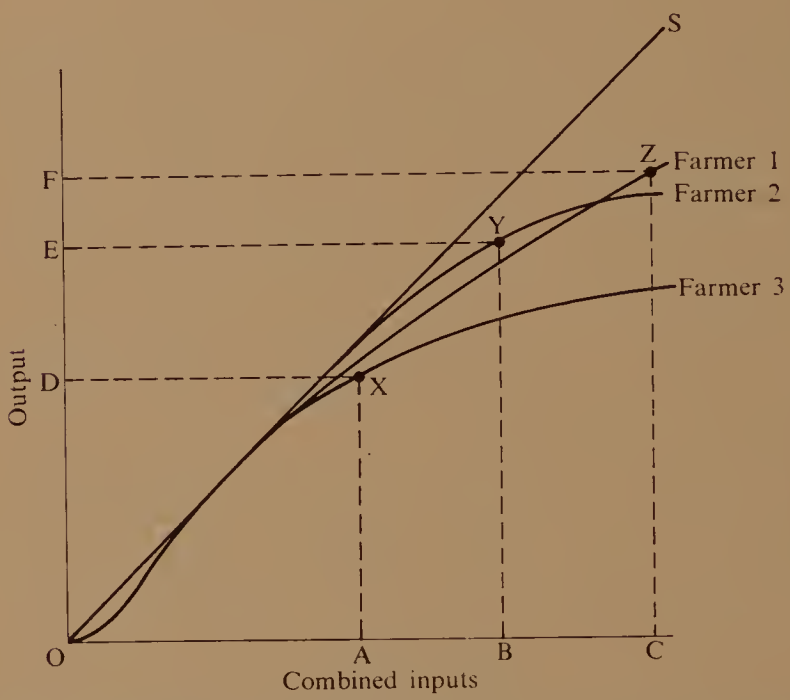


FIG. 8.2. The influence of managerial ability on output. The quantity of all inputs, except management, is measured along the horizontal axis, assuming that the least-cost combination is used throughout. Total product output is measured on the vertical axis. Technically, constant returns to scale might be feasible as shown by the line OS, but in practice, beyond a certain point, output will be limited by the farmer's managerial ability at any given time. From the diagram it appears that farmers 1 and 2 are better managers than farmer 3, although it is impossible to say unambiguously which of farmers 1 and 2 is the more able. The economic optima for farmers 1, 2, and 3 are shown at points Z, Y, and X respectively, representing outputs of OF, OE, and OD units.

straight line. In fact this diagram shows an initial stage of increasing returns to increasing size, which might result from indivisibilities of certain inputs, but this is not important for the present discussion.

As the size of the production unit increases it is likely that the output

achieved in practice by a particular farmer-manager at a particular stage of his career will tend at some point to fall below what is technically feasible, because of limitations of managerial ability and the increasing difficulty of controlling and co-ordinating the larger quantities of all the various inputs. Thus we may expect decreasing returns to size eventually, but the rate of decrease will probably vary from one farmer-manager to another. Fig. 8.2 shows possible production curves for three different farmers at a particular time. Note that it is not always possible to say unambiguously that one farmer is a better manager than another. Thus farmers 1 and 2 both appear to be more efficient than farmer 3 for all levels of input, but whereas the second farmer appears more able than the first at levels of input of OA and OB, their positions are reversed at input OC. This illustrates the earlier suggestion that demonstrated managerial ability at one level of responsibility may not be a reliable guide as to ability at some other level.

Of course, the output achieved from a given combination of productive resources will depend upon climatic conditions and other factors beyond the farmer's control, such as the location of markets. However, in order to simplify the discussion we may assume that these influences are identical for each of the farmers. Like the technical experimenter we may assume that random variations, i.e. good and bad luck, cancel each other out over a number of seasons.

Thus managerial limitations are thought to have a bearing on the size of the production unit, and differences in managerial ability between farmers may provide a possible explanation of the variations in farm sizes observed in practice. Even if they are all pursuing identical objectives, the fact that their production functions differ means that their most favourable scales of operation are likely to differ. Generally, if one manager is unambiguously more able than another, he will probably operate a larger production unit.

For instance, if we assume that the three farmers whose production functions were shown in Fig. 8.2 are all expected-profit maximizers facing identical resource and product prices, we may compare their economic optimum levels of output. Their marginal and average cost curves are shown in Fig. 8.3, those for the first farmer being labelled marginal cost 1 and average cost 1 respectively, those for the second farmer marginal cost 2 and average cost 2, and those for the third marginal cost 3 and average cost 3. The common product price of OP is represented by the horizontal line PP′. Since the marginal cost curve for farmer 2 is further to the right than that of farmer 3, the economic optimum level of output for farmer 2 (OE) is larger than that for farmer 3 (OD). Likewise the

economic optimum level of output for farmer 1 (OF) is larger still. These differing optima are also shown in Fig. 8.2.

From Fig. 8.3 it is apparent that each of the farmers would earn a positive margin over the costs of physical inputs at current price levels, since for each of them average cost is less than the product price at the economic optimum. This margin represents the return to management, and it varies according to individual ability. Note however, that we are

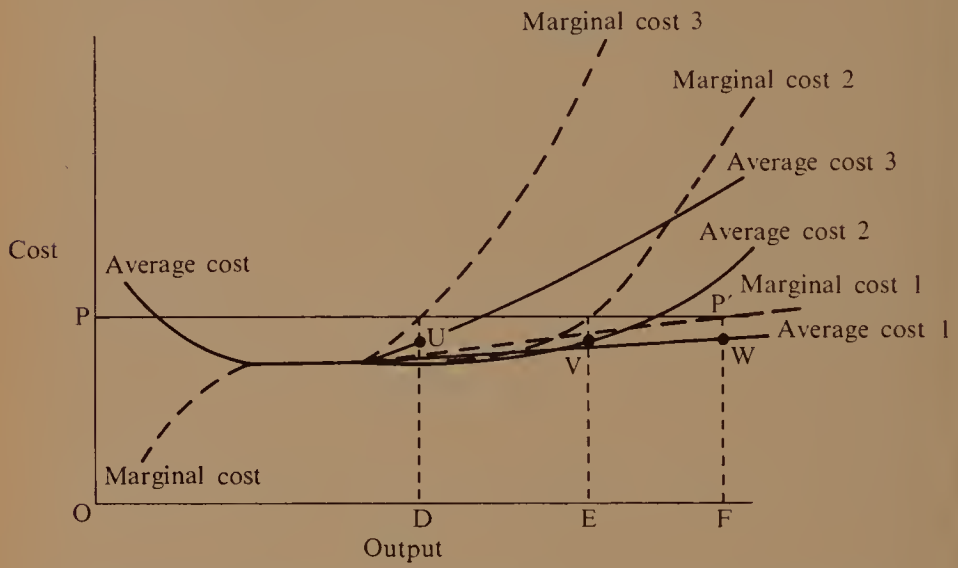


FIG. 8.3. Marginal and average costs. These curves are derived from Fig. 8.2 for each of the 3 farmers. Given a constant marginal revenue of OP represented by the horizontal line PP', it is clear that marginal cost for farmer 1 (marginal cost 1) is equal to marginal revenue at OF units of output, marginal cost 2 is equal to marginal revenue at OE units of output and marginal cost 3 is equal to marginal revenue at OD units of output.

assuming that each manager can earn a similar return whether he is self-employed or working for someone else.

Incidentally, these two diagrams illustrate one of the dangers of trying to estimate production functions or cost functions from cross-sectional comparisons of different farms. The level of output, and of cost per unit of output, on a particular farm depends only in part on the relationship between inputs and output; simultaneously it is influenced by the farmer's choice of input levels, which is a reflection of his decision rule or objective function. Thus an attempt to fit a curve through the points X, Y, and Z in Fig. 8.2 would give a biased and misleading estimate of the production function for any one of the farmers. Similarly a line through U, V, and W in Fig. 8.3 would not represent the relevant cost

function for any farmer. Note, too, that if the surplus over the cost of physical resources is treated as the return to management, and if this is included in the total cost of production, it necessarily follows that average cost per unit of output must equal the product price on all farms regardless of the level of output. Given a constant product price this would imply constant average costs and hence constant returns to scale. However this really tells us nothing about economies and diseconomies of increasing size. Constant returns to scale are the inevitable results of the assumptions made.

Changing managerial limits

The preceding discussion of farm size omits any reference to risk and entrepreneurship because the analysis is static. The achievement of an economic optimum or equilibrium position requires that prices remain stable over a prolonged period, that no capital accumulation takes place and that no technological change occurs. In such circumstances there is no risk and no need for entrepreneurship, but of course it is a totally unrealistic, imaginary situation. In the real and changing world no static, equilibrium distribution of farm sizes is ever achieved. Rather the distribution is constantly changing as some farms grow and others decline. The distribution of farm sizes observed at any point in time may be better explained in terms of differences between farms in their rates of growth rather than in terms of some fixed limitation on their absolute sizes. Furthermore entrepreneurial capacity is likely to be the main constraint on growth, not managerial ability.

This is not to say that the above analysis of the influence of managerial ability on farm size is useless. It may provide a reasonable approximation to the way in which the managerial skill of, say, a pigman or cowman limits the size of unit he can operate. Also it may be viewed as an analysis of the size distribution towards which farms are in the process of adjusting at any point in time. However, before the adjustment is complete the equilibrium size distribution will itself have changed. Thus this so-called 'static theory of the firm' has been described as 'a helpful snapshot description of a system in motion'.[4] Incidentally, it should be noted that, in such continuous dis-equilibrium conditions, there are severe conceptual difficulties in defining what we mean by 'farm size'. Whether it is measured in terms of total output, total cost, or total capital, there are difficulties of aggregation, since the prices ruling at any point in time are

[4] Baumol, W. J. (1962). On the theory of expansion of the firm. *Am. econ. Rev.* **52**, 1078–87.

not equilibrium prices and they are changing over time.[5] Nevertheless we will ignore these difficulties and assume that farm size can be measured satisfactorily in terms of the total capital valuation, including working capital. The growth rate may then be defined as the compound-interest rate of increase in farm size. This of course presumes that the valuation is in real terms and not just current market prices.

There are several reasons for assuming that managerial ability is not an effective limit on growth. First, ability is likely to increase with experience; there is an element of 'learning by doing'.[6] Thus there may be a continuous increase as he grows older in the size of unit an individual is capable of managing. This in itself may provide an incentive for growth. Secondly, managerial responsibility may be delegated in various ways. Even on a small, privately owned farm every person employed probably provides some managerial inputs. Certainly the specialist cowman, pigman, shepherd, or field-foreman does. Beyond this there is the possibility of hiring full-time managers. Hence although the managerial ability of each individual may be an effective limit on the size of enterprise he alone can manage, a farm may be expanded by hiring additional managers along with other inputs. Furthermore as more managers are employed there is more scope for specialization. For instance one manager may concentrate on livestock, another on crops, one may specialize in buying and selling, while another organizes the labour force. As a result each manager is more 'productive' on a larger farm than on a small one where one man is responsible for all these duties. The need for a hierarchy of managers, such as that suggested in Fig. 8.1, may be reduced by developing autonomy of individual units within the whole concern or by laying down rigid rules of behaviour for staff at all levels. The first of these approaches is widely used on commercial farms where individual stockmen are allowed considerable freedom of action in managing semi-independent production units within the whole farm. Similarly, on co-operative settlement schemes, individual settlers, whilst conforming to an over-all production plan, have considerable autonomy. More rigid rules of behaviour are applied on large single-crop schemes like estates and plantations. Thus the managerial limits on size may be penetrated by changing the structure of the organization.

A third reason for assuming that managerial ability is not the main limit on farm growth is that the farmer does not operate on a single

[5] The problems of aggregating capital were discussed in the previous chapter.

[6] See for instance Arrow, K. J. (1962). The economic implications of learning by doing. *Rev. econ. Stud.* **29**, 155–73.

static production function as implied in the preceding analysis. In the real world of changing prices and technological development he may expand his farm by shifting to alternative products and new methods of production. But these shifts involve entrepreneurial, rather than management, decisions. This does not necessarily imply that, to be a successful entrepreneur, a farmer must innovate. He may prefer to imitate activities which his neighbours have found to be profitable, and to avoid their

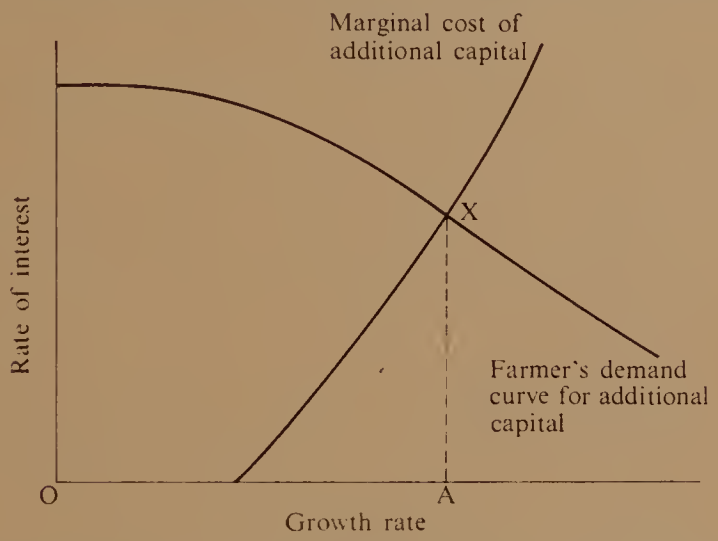


FIG. 8.4. The determinants of growth of the farm business. This looks like Fig. 7.7 (p. 293) but now the horizontal axis measures the percentage growth rate. The farmer's demand curve for additional capital slopes downward more steeply than the incremental internal rate of return curve because demand is limited by his entrepreneurial capacity, his aspirations, and the risk associated with rapid growth. The rising marginal cost curve reflects both the farmer's increasing marginal rate of time-preference and the increasing risk premiums on borrowed funds. Where the curves intersect (point X) determines the growth rate (OA per cent per year). Note that for the percentage growth rate to exceed the percentage rate of interest, as shown in this Figure and the next, savings out of other forms of income must exceed consumption out of interest.

mistakes. However the laggard who fails to respond to changes in the environment, or who responds too late, may find his business declining in size even though it is competently managed.

Entrepreneurship and farm growth

The determination of the growth rate of a farm business may be explained in terms of the capital investment theory discussed in the previous chapter, since growth and change necessarily involve new investment. For this purpose we may use Fig. 8.4, which though similar

to Fig. 7.7 (p. 293) differs from it in certain significant respects. First, the horizontal scale represents the percentage growth rate of the farm, rather than the absolute quantity of new investment. This serves to emphasize what appears to be a general rule, namely that the proportionate rate of change in the size of firms (be it a farm or some other business enterprise) is, to a large extent, independent of absolute size. This finding is often referred to as 'the law of proportionate effect' or 'Gibrat's law' after its originator.[7] Secondly, the 'supply curve' now represents the marginal cost of additional capital raised from both personal savings and sources of credit. As already pointed out, the supply of funds for investment over a given period from both these sources will be limited in total, although the amount made available may be positively related to the interest it will earn. In particular credit agencies are likely to charge a higher risk premium, the greater is the amount borrowed by a particular individual. Thus the rate at which the individual can raise funds for investment sets a limit on the rate of growth of his business.

The third difference is that the farmer's demand curve for additional capital which is shown in this diagram is not necessarily the same as the incremental internal rate of return curve of Fig. 7.7. Indeed, on the basis of the above argument we might now be justified in assuming constant returns to increases in size, in which case the incremental internal rate of return curve would be horizontal. However, the farmer's entrepreneurial capacity will limit the rate at which he can absorb new capital on his farm. For instance there is a limit to the rate at which he can seek out, plan, and establish new productive activities. Linked with this is a limit to the rate at which he can acquire the necessary managerial skills. In the case of an owner-manager time is needed to learn new skills and gain experience. Likewise hired managers become more effective as they gain experience in a particular place of work. Productive efficiency may be impaired if experienced management is diluted too rapidly with fresh blood from outside.

At the same time the farmer may only have limited ambitions. He may be more concerned with having a quiet life or with the 'good husbandry' of his land and livestock than with achieving rapid business growth. It may well be that a man becomes less ambitious in this sense as he grows older. Thus a farmer's attitudes and objectives influence his demand for additional capital. Incidentally, if he objects on moral grounds to being in debt, then clearly this will limit his supply of new capital to what he can save himself.

[7] Gibrat, R. (1931). *Les inégalités économiques*. Sirey, Paris.

Probably the most important factor of all influencing the farmer's demand for investment funds is his attitude to risk. For the reasons given above it seems likely that the faster the rate of growth attempted, the greater will be the attendant risk. In addition, and quite apart from the larger risk premium that may be charged, the farmer's risks will be increased by borrowing more and thereby increasing his gearing. (This principle of increasing risk was discussed in the previous chapter.) Thus if the farmer is at all risk averse, he will borrow and invest less than the maximum possible. However, he would presumably envisage a larger percentage increase in size at a low rate of interest than at a high rate. The demand curve is therefore shown sloping downward to the right.

The expected rate of growth for a particular farmer over a particular period is determined by the intersection of his 'supply curve' and his 'demand curve' for new capital (Fig. 8.4). Thus, for our example farmer, the business is expected to grow by OA per cent per year, but the actual rate of growth achieved is likely to vary from one farmer to another and from year to year. To some extent the differences may be due to random, chance variations. Indeed, some theories of business growth are based on the assumption that *all* differences in proportionate growth rates are due to this cause. However it seems likely that variations between individuals in their entrepreneurial capacities will also have an effect. The personal qualities which would seem to be important in this respect are (1) versatility and adaptability to new products and methods, (2) fund-raising ingenuity, (3) ambition and willingness to take risks, and (4) good judgement. Farmers who are well endowed with these qualities will be able and prepared to offer higher-than-average interest rates for additional capital. At the same time credit agencies are willing to lend on more favourable terms to those they judge to be the better entrepreneurs and managers. Hence their supply and demand curves for new investment will intersect further to the right; their businesses will grow faster than average as shown in Fig. 8.5. Conversely those relatively lacking in the qualities of entrepreneurship will expand their businesses at a less-than-average rate. Indeed, as we have remarked already, some farms may decline in size; their growth rates are negative.

The proportionate rate of business growth for a given individual farmer is likely to change over time as his absolute scale of operation changes. Assessments of his capacity to cope with further growth may well be different at different scales, whether the assessment is that of the farmer himself or that of those who supply credit. Therefore the form of *both* curves in Fig. 8.4 can be expected to change over time, so that the expected growth rate will also change. In addition growth rates are

likely to vary in general according to the relative prosperity, and optimism about the future, of the agricultural industry.

Now it may appear that the more able entrepreneur will both expand his business faster and operate a larger farm than the less able. This would result in a positive relationship between size and growth rate, contrary to Gibrat's Law. However, for other reasons, growth rates may slow down with increasing size. For instance, although management may not set an absolute limit on size, it may restrict growth as the farm gets bigger. Then again, as the farm grows, the farmer may prefer to take advantage of his greater growth potential by increasing his security at the expense of growth.

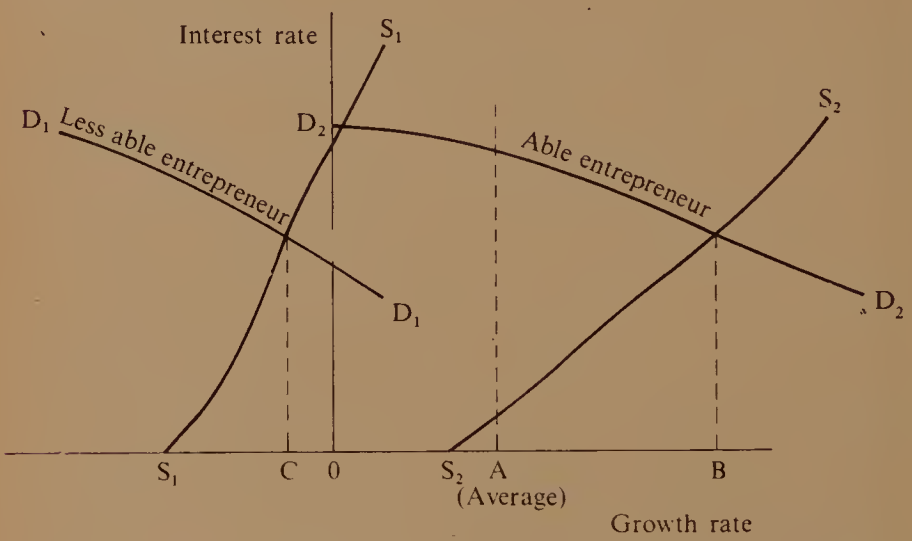


FIG. 8.5. Variation in growth rates. Curves S_1S_1 and D_1D_1 represent the 'supply' and 'demand' for investment funds of an individual who is relatively lacking in entrepreneurial capacity. His business actually declines by OC per cent per year. Curves S_2S_2 and D_2D_2 represent the corresponding 'supply' and 'demand' for investment funds of an able and ambitious entrepreneur, whose business grows by OB per cent per year. Thus at any point in time there is a distribution of growth rates around the mean of OA per cent.

Given that there are differences in growth rate between farms and given that, despite these differences, the proportionate rate of growth is not uniquely related to current size it follows that the size distribution of farms at any point in time will be like that sketched in Fig. 8.6. Furthermore, although individual farms may be growing or declining in size, the over-all size distribution may remain fairly stable in so-called 'steady-state equilibrium'. A frequency distribution such as that shown in Fig. 8.6 is said to be positively skewed, with a few large farms and many

small farms. Generally speaking, the size distributions of farms met with in practice are of this form, thus lending support to the theory. It is less easy to explain such size distributions with the static theory discussed earlier, unless managerial abilities are assumed to be similarly distributed.

However, the dynamic theory developed above is incomplete since it does not explain how new farms are established or 'born' and how other farms disappear or 'die'. A complete explanation is not attempted here; indeed it would need to be quite complicated, involving economic, social, and political factors. However, some general comments may be made.

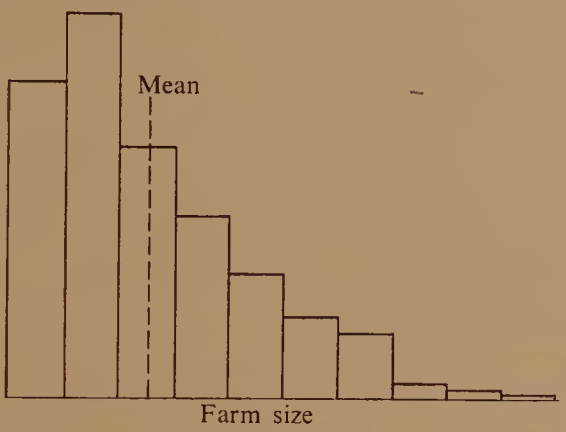


FIG. 8.6. The size distribution of farms. Distributions of a similar, skewed, form are found whether size is measured in hectares of land, number of workers or current capital valuation. Such a size distribution will result from a reasonably symmetrical distribution of proportionate growth rates which are not uniquely related to current size.

First, it seems likely that most newly established farms will be small. Relatively few will be established on a large scale. Secondly, most deaths are likely to occur among the smaller farms also. Indeed, because of the indivisibilities of certain inputs there may be a 'minimum efficient size', below which average costs of production must exceed those on larger farms. Farms of less than the minimum efficient size are particularly vulnerable. However some large farms may be broken up occasionally, for instance when on retirement a man divides his property up among his heirs.

Finally, the birth and death rates and hence the total number of farm businesses may vary according to the relative prosperity of the agricultural industry. When agricultural prices are rising in relation to those in other industries, new capital is likely to flow into agriculture and one might expect more farms to be established. However, the average growth rate of existing farms is likely to be raised in such conditions and as a

result the entrepreneurial income on a farm of any given size may rise very little. Thus the increase in output per farm is likely to be greater than the increase in the number of farms. Similarly a relative decline in agricultural prices may cause a fall in growth rates, or even in average farm size rather than a reduction in numbers of farms.

Sharing entrepreneurial responsibility

In developing this theory we have assumed, for simplicity, that the farmer is solely responsible for entrepreneurial decision making. This is not necessarily the case. Even on a small, privately owned farm, a so-called 'individual proprietorship', it is likely that the farmer's wife and possibly his children may be involved in the decision-making process and share in the risks. Furthermore he may share the entrepreneurial function with suppliers of his inputs and purchasers of his outputs. For instance under a system of share-cropping, both landlord and tenant contribute to entrepreneurial decision making and share the risks. When a farmer arranges an advance contract for the sale of his produce, he is in effect unloading a part of the entrepreneurial function and some of the risk onto the buyer. With vertical integration, the location of entrepreneurial decision making and the limits of the business become blurred. For example, where a poultry feed company supplies chicks and feed to the farmer and pays him a guaranteed price for the grown bird, the poultry unit may be viewed either as a part of the farm or as a part of the poultry feed company.

Partnerships and co-operatives are formal, or informal, institutions for sharing the entrepreneurial task. An ordinary partnership is an arrangement whereby two or more persons jointly own, in specific proportions, the capital of a business and, in theory, jointly run it. However, it can happen that one or more of the partners may take no part in the running of the business. Nonetheless such 'sleeping partners' share the risk. Generally the residual income, after all contractual costs have been met, is shared in accordance with the amount of capital contributed. However, in the event of other partners failing to honour their obligations, an individual partner will be responsible for all debts to the limit of his own personal fortune no matter how held and regardless of his own nominal share in the business. Thus each partner shares the risk in the fullest way possible.

Although various arrangements are possible, an individual member of a co-operative generally cannot be held responsible for all debts; his liability is limited to the extent of his own contribution to the pool of capital. A co-operative is formed for the mutual benefit of the members,

each of whom contributes roughly the same amount of capital. Any surplus which may be earned, if not used for the maintenance or development of the co-operative, is returned to members in proportion to the amount of business done and *not* on the basis of the amount of capital contributed. In principle, co-operatives are run democratically with each member having a single vote in planning the operations. However, particularly in the larger co-operatives, entrepreneurial decisions may be taken by hired managers. Probably the most democratically run, large-scale agricultural production units are the co-operative settlements in Israel.

In North America and Western Europe, manufacturing industry is typically organized into joint stock, limited liability companies. Such a company is owned by the 'shareholders', usually large in number, who contribute capital by buying shares. These shareholders are represented by the 'directors' whom they elect to administer their capital. Any profit which results after the directors' salaries and, of course, other costs, have been met is divided among the shareholders in accordance with the amount of capital invested. The force of the word 'limited' is that should the company fail and run into debt, the maximum amount the shareholder can be expected to lose is the price he paid for his shares.

Once this principle of limited liability is given the force of law, the most important limitation on the growth of a business firm is removed since expansion no longer depends upon the personal financial position of the owners. So long as owners are personally liable for the actions of their agents as well as for any debts that may be incurred they will want to be intimately involved in all entrepreneurial decision making. With strictly limited liability the shareholder is able to delegate authority entirely and take little or no part in the running of the business. New equity capital may be raised by issuing new shares to anyone who cares to buy them. Thus limited liability companies may grow to a much larger size than other forms of privately owned business.

Although in theory the directors are the representatives of the shareholders and might therefore be expected to make the entrepreneurial decisions, many writers believe that 'entrepreneurship in the modern corporation has been taken over by transcendent (hired) management, whose functions differ in kind from those of the traditional subordinate or "mere manager". These, it is argued, 'can wield considerable power without necessarily holding equity, sharing profits or carrying risks.'[8]

[8] See Marris, R. (1964). *The economic theory of 'managerial' capitalism.* Macmillan, London.

Thus their objectives may differ from those of the traditional entrepreneur. It is suggested for instance that they may be concerned with maximizing sales or the growth of sales subject to the provision of an acceptable return being paid on share capital. The promotion prospects of a manager may be more closely related to sales than to profit.

Limited liability companies are not common in agriculture, however. This may be an accident of history but it may alternatively be due to the relative inefficiency of very large businesses in agriculture. In comparison with other industries agricultural production must be spread over a comparatively large area and is relatively labour-intensive. Thus the problems of managing and supervising labour may increase with increasing size more rapidly in agriculture than in other industries.

Summary

1. A distinction is drawn between management, the administrative and co-ordinating function, and entrepreneurship, the innovating and risk-taking function.

2. Managerial ability may set a limit on the most favourable size of production unit an individual can control at a given point in time.

3. However, the managerial limit may recede as an individual gains experience, or as more managers are hired or as new products and processes are found.

4. Farm growth is limited by the farmer's entrepreneurial capacity, his versatility, his ability to raise funds on favourable terms, and his ambition and willingness to take risks and his judgement.

5. Despite differences between individuals, the growth rate does not bear any strong relationship to the current farm size. As a result the size distribution of farms at any point in time is positively skewed.

6. Various alternative organizational structures enable the entrepreneurial task to be shared.

Suggestions for further reading

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Appendix: Tables

TABLE A.1

Future value (B_t) of £1 in t years. $B_t = (1 + i)^t$

(t)	Rate of interest (i) per cent														
	2·00	4·00	6·00	8·00	10·00	12·00	14·00	16·00	18·00	20·00	22·00	24·00	26·00	28·00	30·00
1	1·02	1·04	1·06	1·08	1·10	1·12	1·14	1·16	1·18	1·20	1·22	1·24	1·26	1·28	1·30
2	1·04	1·08	1·12	1·17	1·21	1·25	1·30	1·35	1·39	1·44	1·49	1·54	1·59	1·64	1·69
3	1·06	1·12	1·19	1·26	1·33	1·40	1·48	1·56	1·64	1·73	1·82	1·91	2·00	2·10	2·20
4	1·08	1·17	1·26	1·36	1·46	1·57	1·69	1·81	1·94	2·07	2·22	2·36	2·52	2·68	2·86
5	1·10	1·22	1·34	1·47	1·61	1·76	1·93	2·10	2·29	2·49	2·70	2·93	3·18	3·44	3·71
6	1·13	1·27	1·42	1·59	1·77	1·97	2·19	2·44	2·70	2·99	3·30	3·64	4·00	4·40	4·83
7	1·15	1·32	1·50	1·71	1·95	2·21	2·50	2·83	3·19	3·58	4·02	4·51	5·04	5·63	6·27
8	1·17	1·37	1·59	1·85	2·14	2·48	2·85	3·28	3·76	4·30	4·91	5·59	6·35	7·21	8·16
9	1·20	1·42	1·69	2·00	2·36	2·77	3·25	3·80	4·44	5·16	5·99	6·93	8·00	9·22	10·60
10	1·22	1·48	1·79	2·16	2·59	3·11	3·71	4·41	5·23	6·19	7·30	8·59	10·09	11·81	13·79
11	1·24	1·54	1·90	2·33	2·85	3·48	4·23	5·12	6·18	7·43	8·91	10·66	12·71	15·11	17·92
12	1·27	1·60	2·01	2·52	3·14	3·90	4·82	5·94	7·29	8·92	10·87	13·21	16·01	19·34	23·30
13	1·29	1·67	2·13	2·72	3·45	4·36	5·49	6·89	8·60	10·70	13·26	16·39	20·18	24·76	30·29
14	1·32	1·73	2·26	2·94	3·80	4·89	6·26	7·99	10·15	12·84	16·18	20·32	25·42	31·69	39·37
15	1·35	1·80	2·40	3·17	4·18	5·47	7·14	9·27	11·97	15·41	19·74	25·20	32·03	40·56	51·19
16	1·37	1·87	2·54	3·43	4·59	6·13	8·14	10·75	14·13	18·49	24·09	31·24	40·36	51·92	66·54
17	1·40	1·95	2·69	3·70	5·05	6·87	9·28	12·47	16·67	22·19	29·38	38·74	50·85	66·46	86·50
18	1·43	2·03	2·85	4·00	5·56	7·69	10·58	14·46	19·67	26·62	35·85	48·04	64·07	85·07	112·46
19	1·46	2·11	3·03	4·32	6·12	8·61	12·06	16·78	23·21	31·95	43·74	59·57	80·73	108·89	146·19
20	1·49	2·19	3·21	4·66	6·73	9·65	13·74	19·46	27·39	38·34	53·36	73·86	101·72	139·38	190·05
21	1·52	2·28	3·40	5·03	7·40	10·80	15·67	22·57	32·32	46·01	65·10	91·59	128·17	178·41	247·06
22	1·55	2·37	3·60	5·44	8·14	12·10	17·86	26·19	38·14	55·21	79·42	113·57	161·49	228·36	321·18
23	1·58	2·46	3·82	5·87	8·95	13·55	20·36	30·38	45·01	66·25	96·89	140·83	203·48	292·30	417·54
24	1·61	2·56	4·05	6·34	9·85	15·18	23·21	35·24	53·11	79·50	118·21	174·63	256·39	374·14	542·80
25	1·64	2·67	4·29	6·85	10·83	17·00	26·46	40·87	62·67	95·40	144·21	216·54	323·05	478·90	705·64
26	1·67	2·77	4·55	7·40	11·92	19·04	30·17	47·41	73·95	114·48	175·94	268·51	407·04	613·00	917·33
27	1·71	2·88	4·82	7·99	13·11	21·32	34·39	55·00	87·26	137·37	214·64	332·95	512·87	784·64	1192·53
28	1·74	3·00	5·11	8·63	14·42	23·88	39·20	63·80	102·97	164·84	261·86	412·86	646·21	1004·34	1550·29
29	1·78	3·12	5·42	9·32	15·86	26·75	44·69	74·01	121·50	197·81	319·47	511·95	814·23	1285·55	2015·38
30	1·81	3·24	5·74	10·06	17·45	29·96	50·95	85·85	143·37	237·38	389·76	634·82	1025·93	1645·50	2620·00

Note. More detailed figures are given in Lawson, G. H. and Windle, D. W. (1965). *Tables of discounted cash flow, annuity, sinking fund, compound interest and annual capital charge calculations.* Oliver and Boyd, Edinburgh.

TABLE A.2

Present value (P) of £1 due in t years. $P = \frac{1}{(1+i)^t}$

(t)	Rate of interest (i) per cent													
	2·00	4·00	6·00	8·00	10·00	12·00	14·00	16·00	18·00	20·00	22·00	24·00	26·00	28·00	30·00
1	0·9804	0·9615	0·9434	0·9259	0·9091	0·8929	0·8772	0·8621	0·8475	0·8333	0·8197	0·8065	0·7937	0·7813	0·7692
2	0·9612	0·9246	0·8900	0·8573	0·8264	0·7972	0·7695	0·7432	0·7182	0·6944	0·6719	0·6504	0·6299	0·6104	0·5917
3	0·9423	0·8890	0·8396	0·7938	0·7513	0·7118	0·6750	0·6407	0·6086	0·5787	0·5507	0·5245	0·4999	0·4768	0·4552
4	0·9238	0·8548	0·7921	0·7350	0·6830	0·6355	0·5921	0·5523	0·5158	0·4823	0·4514	0·4230	0·3968	0·3725	0·3501
5	0·9057	0·8219	0·7473	0·6806	0·6209	0·5674	0·5194	0·4761	0·4371	0·4019	0·3700	0·3411	0·3149	0·2910	0·2693
6	0·8880	0·7903	0·7050	0·6302	0·5645	0·5066	0·4556	0·4104	0·3704	0·3349	0·3033	0·2751	0·2499	0·2274	0·2072
7	0·8706	0·7599	0·6651	0·5835	0·5132	0·4523	0·3996	0·3538	0·3139	0·2791	0·2486	0·2218	0·1983	0·1776	0·1594
8	0·8535	0·7307	0·6274	0·5403	0·4665	0·4039	0·3506	0·3050	0·2660	0·2326	0·2038	0·1789	0·1574	0·1388	0·1226
9	0·8368	0·7026	0·5919	0·5002	0·4241	0·3606	0·3075	0·2630	0·2255	0·1938	0·1670	0·1443	0·1249	0·1084	0·0943
10	0·8203	0·6756	0·5584	0·4632	0·3855	0·3220	0·2697	0·2267	0·1911	0·1615	0·1369	0·1164	0·0992	0·0847	0·0725
11	0·8043	0·6496	0·5268	0·4289	0·3505	0·2875	0·2366	0·1954	0·1619	0·1346	0·1122	0·0938	0·0787	0·0662	0·0558
12	0·7885	0·6246	0·4970	0·3971	0·3186	0·2567	0·2076	0·1685	0·1372	0·1122	0·0920	0·0757	0·0625	0·0517	0·0429
13	0·7730	0·6006	0·4688	0 3677	0·2897	0·2292	0·1821	0·1452	0·1163	0·0935	0·0754	0·0610	0·0496	0·0404	0·0330
14	0·7579	0·5775	0·4423	0·3405	0·2633	0·2046	0·1597	0·1252	0·0985	0·0779	0·0618	0·0492	0·0393	0·0316	0·0254
15	0·7430	0·5553	0·4173	0·3152	0·2394	0·1827	0·1401	0·1079	0·0835	0·0649	0·0507	0·0397	0·0312	0·0247	0·0195
16	0·7284	0·5339	0·3936	0·2919	0·2176	0·1631	0·1229	0·0930	0·0708	0·0541	0·0415	0·0320	0·0248	0·0193	0·0150
17	0·7142	0·5134	0·3714	0·2703	0·1978	0·1456	0·1078	0·0802	0·0600	0·0451	0·0340	0·0258	0·0197	0·0150	0·0116
18	0·7002	0·4936	0·3503	0·2502	0·1799	0·1300	0·0946	0·0691	0·0508	0·0376	0·0279	0·0208	0·0156	0·0118	0·0089
19	0·6864	0·4746	0·3305	0·2317	0·1635	0·1161	0·0829	0·0596	0·0431	0·0313	0·0229	0·0168	0·0124	0·0092	0·0068
20	0·6730	0·4564	0·3118	0·2145	0·1486	0·1037	0·0728	0·0514	0·0365	0·0261	0·0187	0·0135	0·0098	0·0072	0·0053
21	0·6598	0·4388	0·2942	0·1987	0·1351	0·0926	0·0638	0·0443	0·0309	0·0217	0·0154	0·0109	0·0078	0·0056	0·0040
22	0·6468	0·4220	0·2775	0·1839	0·1228	0·0826	0·0560	0·0382	0·0262	0·0181	0·0126	0·0088	0·0062	0·0044	0·0031
23	0·6342	0·4057	0·2618	0·1703	0·1117	0·0738	0·0491	0·0329	0·0222	0·0151	0·0103	0·0071	0·0049	0·0034	0·0024
24	0·6217	0·3901	0·2470	0·1577	0·1015	0·0659	0·0431	0·0284	0·0188	0·0126	0·0085	0·0057	0·0039	0·0027	0·0018
25	0·6095	0·3751	0·2330	0·1460	0·0923	0·0588	0·0378	0·0245	0·0160	0·0105	0·0069	0·0046	0·0031	0·0021	0·0014
26	0·5976	0·3607	0·2198	0·1352	0·0839	0·0525	0·0331	0·0211	0·0135	0·0087	0·0057	0·0037	0·0025	0·0016	0·0011
27	0·5859	0·3468	0·2074	0·1252	0·0763	0·0469	0·0291	0·0182	0·0115	0·0073	0·0047	0·0030	0·0019	0·0013	0·0008
28	0·5744	0·3335	0·1956	0·1159	0·0693	0·0419	0·0255	0·0157	0·0097	0·0061	0·0038	0·0024	0·0015	0·0010	0·0006
29	0·5631	0·3207	0·1846	0·1073	0·0630	0·0374	0·0224	0·0135	0·0082	0·0051	0·0031	0·0020	0·0012	0·0008	0·0005
30	0·5521	0·3083	0·1741	0·0994	0·0573	0·0334	0·0196	0·0116	0·0070	0·0042	0·0026	0·0016	0·0010	0·0006	0·0004

TABLE A.3

Future value (TA) at t years of an annuity of £1. $(TA) = \dfrac{(1+i)^t - 1}{i}$

(t)	Rate of interest (i) per cent														
	2·00	4·00	6·00	8·00	10·00	12·00	14·00	16·00	18·00	20·00	22·00	24·00	26·00	28·00	30·00
1	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00
2	2·02	2·04	2·06	2·08	2·10	2·12	2·14	2·16	2·18	2·20	2·22	2·24	2·26	2·28	2·30
3	3·06	3·12	3·18	3·25	3·31	3·37	3·44	3·51	3·57	3·64	3·71	3·78	3·85	3·92	3·99
4	4·12	4·25	4·37	4·51	4·64	4·78	4·92	5·07	5·22	5·37	5·52	5·68	5·85	6·02	6·19
5	5·20	5·42	5·64	5·87	6·11	6·35	6·61	6·88	7·15	7·44	7·74	8·05	8·37	8·70	9·04
6	6·31	6·63	6·98	7·34	7·72	8·12	8·54	8·98	9·44	9·93	10·44	10·98	11·54	12·14	12·76
7	7·43	7·90	8·39	8·92	9·49	10·09	10·73	11·41	12·14	12·92	13·74	14·62	15·55	16·53	17·58
8	8·58	9·21	9·90	10·64	11·44	12·30	13·23	14·24	15·33	16·50	17·76	19·12	20·59	22·16	23·86
9	9·75	10·58	11·49	12·49	13·58	14·78	16·09	17·52	19·09	20·80	22·67	24·71	26·94	29·37	32·01
10	10·95	12·01	13·18	14·49	15·94	17·55	19·34	21·32	23·52	25·96	28·66	31·64	34·94	38·59	42·62
11	12·17	13·49	14·97	16·65	18·53	20·65	23·04	25·73	28·76	32·15	35·96	40·24	45·03	50·40	56·41
12	13·41	15·03	16·87	18·98	21·38	24·13	27·27	30·85	34·93	39·58	44·87	50·89	57·74	65·51	74·33
13	14·68	16·63	18·88	21·50	24·52	28·03	32·09	36·79	42·22	48·50	55·75	64·11	73·75	84·85	97·63
14	15·97	18·29	21·02	24·21	27·97	32·39	37·58	43·67	50·82	59·20	69·01	80·50	93·93	109·61	127·91
15	17·29	20·02	23·28	27·15	31·77	37·28	43·84	51·66	60·97	72·04	85·19	100·82	119·35	141·30	167·29
16	18·64	21·82	25·67	30·32	35·95	42·75	50·98	60·93	72·94	87·44	104·93	126·01	151·38	181·87	218·47
17	20·01	23·70	28·21	33·75	40·54	48·88	59·12	71·67	87·07	105·93	129·02	157·25	191·73	233·79	285·01
18	21·41	25·65	30·91	37·45	45·60	55·75	68·39	84·14	103·74	128·12	158·40	195·99	242·59	300·25	371·52
19	22·84	27·67	33·76	41·45	51·16	63·44	78·97	98·60	123·41	154·74	194·25	244·03	306·66	385·32	483·97
20	24·30	29·78	36·79	45·76	57·27	72·05	91·02	115·38	146·63	186·69	237·99	303·60	387·39	494·21	630·17
21	25·78	31·97	39·99	50·42	64·00	81·70	104·77	134·84	174·02	225·03	291·35	377·46	489·11	633·59	820·22
22	27·30	34·25	43·39	55·46	71·40	92·50	120·44	157·41	206·34	271·03	356·44	469·06	617·28	812·00	1067·28
23	28·84	36·62	47·00	60·89	79·54	104·60	138·30	183·60	244·49	326·24	435·86	582·63	778·77	1040·36	1388·46
24	30·42	39·08	50·82	66·76	88·50	118·16	158·66	213·98	289·49	392·48	532·75	723·46	982·25	1332·66	1806·00
25	32·03	41·65	54·86	73·11	98·35	133·33	181·87	249·21	342·60	471·98	650·96	898·09	1238·64	1706·80	2348·80
26	33·67	44·31	59·16	79·95	109·18	150·33	208·33	290·09	405·27	567·38	795·17	1114·63	1561·68	2185·71	3054·44
27	35·34	47·08	63·71	87·35	121·10	169·37	238·50	337·50	479·22	681·85	971·10	1383·15	1968·72	2798·71	3971·78
28	37·05	49·97	68·53	95·34	134·21	190·70	272·89	392·50	566·48	819·22	1185·74	1716·10	2481·59	3583·34	5164·31
29	38·79	52·97	73·64	103·97	148·63	214·58	312·09	456·30	669·45	984·07	1447·61	2128·96	3127·80	4587·68	6714·60
30	40·57	56·08	79·06	113·28	164·49	241·33	356·79	530·31	790·95	1181·88	1767·08	2640·92	3942·03	5873·23	8729·99

Table A.4

Present value (PA) of an annuity of £1 payable for t future years. $(PA) = \frac{(1+i)^t - 1}{i(1+i)^t}$

(t)	Rate of interest (i) per cent														
	2·00	4·00	6·00	8·00	10·00	12·00	14·00	16·00	18·00	20·00	22·00	24·00	26·00	28·00	30·00
1	0·9804	0·9615	0·9434	0·9259	0·9091	0·8929	0·8772	0·8621	0·8475	0·8333	0·8197	0·8065	0·7937	0·7813	0·7692
2	1·9416	1·8861	1·8334	1·7833	1·7355	1·6901	1·6467	1·6052	1·5656	1·5278	1·4915	1·4568	1·4235	1·3916	1·3609
3	2·8839	2·7751	2·6730	2·5771	2·4869	2·4018	2·3216	2·2459	2·1743	2·1065	2·0422	1·9813	1·9234	1·8684	1·8161
4	3·8077	3·6299	3·4651	3·3121	3·1699	3·0373	2·9137	2·7982	2·6901	2·5887	2·4936	2·4043	2·3202	2·2410	2·1662
5	4·7135	4·4518	4·2124	3·9927	3·7908	3·6048	3·4331	3·2743	3·1272	2·9906	2·8636	2·7454	2·6351	2·5320	2·4356
6	5·6014	5·2421	4·9173	4·6229	4·3553	4·1114	3·8887	3·6847	3·4976	3·3255	3·1669	3·0205	2·8850	2·7594	2·6427
7	6·4720	6·0021	5·5824	5·2064	4·8684	4·5638	4·2883	4·0386	3·8115	3·6046	3·4155	3·2423	3·0833	2·9370	2·8021
8	7·3255	6·7327	6·2098	5·7466	5·3349	4·9676	4·6389	4·3436	4·0776	3·8372	3·6193	3·4212	3·2407	3·0758	2·9247
9	8·1622	7·4353	6·8017	6·2469	5·7590	5·3282	4·9464	4·6065	4·3030	4·0310	3·7863	3·5655	3·3657	3·1842	3·0190
10	8·9826	8·1109	7·3601	6·7101	6·1446	5·6502	5·2161	4·8332	4·4941	4·1925	3·9232	3·6819	3·4648	3·2689	3·0915
11	9·7868	8·7605	7·8869	7·1390	6·4951	5·9377	5·4527	5·0286	4·6560	4·3271	4·0354	3·7757	3·5435	3·3351	3·1473
12	10·5753	9·3851	8·3838	7·5361	6·8137	6·1944	5·6603	5·1971	4·7932	4·4392	4·1274	3·8514	3·6059	3·3868	3·1903
13	11·3484	9·9856	8·8527	7·9038	7·1034	6·4235	5·8424	5·3423	4·9095	4·5327	4·2028	3·9124	3·6555	3·4272	3·2233
14	12·1062	10·5631	9·2950	8·2442	7·3667	6·6282	6·0021	5·4675	5·0081	4·6106	4·2646	3·9616	3·6949	3·4587	3·2487
15	12·8493	11·1184	9·7122	8·5595	7·6061	6·8109	6·1422	5·5755	5·0916	4·6755	4·3152	4·0013	3·7261	3·4834	3·2682
16	13·5777	11·6523	10·1059	8·8514	7·8237	6·9740	6·2651	5·6685	5·1624	4·7296	4·3567	4·0333	3·7509	3·5026	3·2832
17	14·2919	12·1657	10·4773	9·1216	8·0216	7·1196	6·3729	5·7487	5·2223	4·7746	4·3908	4·0591	3·7705	3·5177	3·2948
18	14·9920	12·6593	10·8276	9·3719	8·2014	7·2497	6·4674	5·8178	5·2732	4·8122	4·4187	4·0799	3·7861	3·5294	3·3037
19	15·6785	13·1339	11·1581	9·6036	8·3649	7·3658	6·5504	5·8775	5·3162	4·8435	4·4415	4·0967	3·7985	3·5386	3·3105
20	16·3514	13·5903	11·4699	9·8181	8·5136	7·4694	6·6231	5·9288	5·3527	4·8696	4·4603	4·1103	3·8083	3·5458	3·3158
21	17·0112	14·0292	11·7641	10·0168	8·6487	7·5620	6·6870	5·9731	5·3837	4·8913	4·4756	4·1212	3·8161	3·5514	3·3198
22	17·6580	14·4511	12·0416	10·2007	8·7715	7·6446	6·7429	6·0113	5·4099	4·9094	4·4882	4·1300	3·8223	3·5558	3·3230
23	18·2922	14·8568	12·3034	10·3711	8·8832	7·7184	6·7921	6·0442	5·4321	4·9245	4·4985	4·1371	3·8273	3·5592	3·3254
24	18·9139	15·2470	12·5504	10·5288	8·9847	7·7843	6·8351	6·0726	5·4509	4·9371	4·5070	4·1428	3·8312	3·5619	3·3272
25	19·5235	15·6221	12·7834	10·6748	9·0770	7·8431	6·8729	6·0971	5·4669	4·9476	4·5139	4·1474	3·8342	3·5640	3·3286
26	20·1210	15·9828	13·0032	10·8100	9·1609	7·8957	6·9061	6·1182	5·4804	4·9563	4·5196	4·1511	3·8367	3·5656	3·3297
27	20·7069	16·3296	13·2105	10·9352	9·2372	7·9426	6·9352	6·1364	5·4919	4·9636	4·5243	4·1542	3·8387	3·5669	3·3305
28	21·2813	16·6631	13·4062	11·0511	9·3066	7·9844	6·9607	6·1520	5·5016	4·9697	4·5281	4·1566	3·8402	3·5679	3·3312
29	21·8444	16·9837	13·5907	11·1584	9·3696	8·0218	6·9830	6·1656	5·5098	4·9747	4·5312	4·1585	3·8414	3·5687	3·3317
30	22·3965	17·2920	13·7648	11·2578	9·4269	8·0552	7·0027	6·1772	5·5168	4·9789	4·5338	4·1601	3·8424	3·5693	3·3321

Index

ability, managerial 5, 322, 325–38; *see also* management
activities,
  combinations of, 40–2, 49
  substitution of, 40–9
activity (linear programming), 36, 60–1
  disposal, 59(n), 62
  dominated, 44, 302
  least-cost, 88
advance contracts, 183
aggregation, 5, 247(n)
  of capital, 290, 329–30
agricultural
  policy, 258–9
  sector, 253–9
Allen, W., 200(n)
Anderson, J. R., 159(n)
annuity, 282, 311–13
  equivalent, 282–3
  tables, 344–5
Arrow, K. J., 330(n)
assets, 260; *see also* capital
  terminal, 301–6
attitudes and objectives, 292, 332
  to risk, 161, 333
average cost, 84–8, 90–2, 95–6
average product, 8–11, 26–8; *see also* productivity
  per person, 196–8, 250–3
averages, fallacy of, 145–51, 188
averaging out, 147; *see also* tree diagrams
aversion to risk, 151–3, 237

backward-bending supply curve, 244
backwards induction, 147; *see also* dynamic programming
Bailey, M. J., 278(n)
Barber, D., 75(n)
Baumol, W. J., 129, 329(n)
basic (feasible) solution, 62–72, 112; *see also* corner points, linear programming
Bayes, T., 172(n)
Becker, G. S., 259
behaviour, rational, 75
Behrens, W. W. III, 195(n)
biased innovations, 201; *see also* technological change
Bierman, H., 320
Blaug, M., 320
borrowing, 265, 295–9; *see also* capital, interest
Boulding, K. E., 72, 252(n), 290(n)
boundary optima, 98; *see also* corner optima
  production possibility, 24, 54–9, 110, 112; *see also* production possibility curve
  cost curve as a, 94
Boyd, D. A., 6(n), 7(n)
Boyer, R. S., 338
Bressler, R. G., 231
Bronfenbrenner, M., 338

calculus, differential, 28(n), 31(n), 32(n), 33(n), 35(n), 118(n), 120(n), 122(n), 125(n), 192(n)
capital, 3, 260–320
  aggregation, 290, 329–30
  and innovation, 204, 206
  and money, 3–4, 261; *see also* working capital
  circulating, 262
  demand for, 287–8
  formation, 264
  gains and losses, 228, 290, 323
  indivisibility, 300
  investment, 263–7
  landlords, 222; *see also* leasehold tenure
  measurement, 290
  negative 278; *see also* double switching
  rationing, 299
  return on, 261, 263
  reversing 291; *see also* double switching
  theoretical difficulties with, 288–91
  valuation, 289-90
  working, 224, 262
capitalized value, 273
Carter, H. O., 172(n)

case I and case II situations, 251–3
Casey, H., 72
cash flows, 271–5, 311–13; *see also* compounding, discounting
casual hired labour, 233, 238–9, 242
certainty equivalent, 161, 166; *see also* risk aversion
Champernowne, D. G., 290(n)
change of technique, 202; *see also* technological change
changing managerial limits, 329–31; *see also* farm size
Cheung, S. N. S., 231
Chisholm, M , 231
choice of products, 124–6; *see also* product – product relationships, linear programming
choices under risk, 159–65; *see also* expected utility
circulating capital, 262; *see also* working capital
climate and land-use, 207–14
Cocks, K. D., 320
collateral security, 298; *see also* credit
combinations,
  enterprise (product), 21–5
  factor (input, resource), 15–21
  infeasible, 45
  least-cost, 81–7, 94, 97, 120–1
  linear, 41
  maximum profit (revenue), 108–16
  of activities, 40–2, 44–8, 51(n)
communal rights, 230; *see also* land tenure
comparative advantage, 212–14, 217–18; *see also* enterprise combinations, opportunity cost
competing enterprises, 21, 22, 25, 111; *see also* enterprise combinations
competition, pure, 75, 131, 295; *see also* constant prices
complementary
  enterprises, 25, 111; *see also* enterprise combinations
  factors/inputs/resources, 39
  projects, 307
compound interest, 268
compounding, 267–71, 310–11; *see also* future value, interest
  cash flows, 272
  tables, 342
  ncavity
coof production-possibility boundary, 25, 58; *see also* increasing rate of product transformation
concavity *cont.*—
  of response curve, 11, 26; *see also* diminishing marginal returns
conditional
  expectations, 145
  probability, 138–45, 185
conservation, 199, 226
consumer capital, 260–1
constant
  average cost, 87
  factor proportions, 19, 21, 31; *see also* scale
  marginal product, 21, 32, 33(n), 37; *see also* linear production relationships
  marginal revenue, 94, 96
  prices, 75–6
  rate of technical substitution, 19, 43
  returns to scale, 20–1, 31–3, 326, 329; *see also* constant average cost, linear production relationships, product exhaustion
constraints, 36–9, 60–72
  and rising marginal cost, 97–9
  and shadow prices, 103–4, 123–5
  and the feasible region, 45–54, 75
  and the production-possibility boundary, 54–9
  and usable yield, 148–51
  effective, 36
  land, 200, 210–11
  linear-programming, 126–9
  on consumption, borrowing and lending, 303–5
  on funds, 301
  seasonal labour, 237–9
  survival, 162–4
  treating objectives as, 161
consumption, 74, 255, 302–5, 318–19; *see also* subsistence, survival needs
continuous
  curves, 8, 12, 16–17, 24–5; *see also* production function
  optimality, 318; *see also* multiperiod planning
  variable, 135–6
contracts, advance, 183
contractual costs, 93, 227, 323; *see also* entrepreneurship, fixed costs
convexity of isoquants, 18, 44–5; *see also* diminishing rate of technical substitution
Cooke, G. W., 7(n)
co-operatives, 336–7

corner
  optima, 98–9
  points, 50–1, 59, 112; *see also* basic feasible solution
correlation coefficient, 156(n), 189–90; *see also* covariance
cost,
  average, 84–8, 90–2, 95–6
  -benefit analysis, 260(n)
  contractual, 93, 227, 323; *see also* entrepreneurship, fixed costs
  curve, 94–6
  fixed, 90–3
  function, 84, 95, 98, 328–9
  marginal, 87, 93–8, 118–19, 121–3, 126
  marginal factor, 77(n)
  of acquisition, 102
  opportunity, 23, 47, 54, 74, 107–11; *see also* shadow price
  schedule, 94–5, 99
  terminal, 215; *see also* transport
  total, 74, 76–80, 84–5
  transfer, 215–16; *see also* transport
  transport, 214–16
  unit factor, 76–9, 118, 122
  variable, 95–100
costs and the economic optimum, 94–9
covariance, 155–8, 189–92
Cowling, K., 75(n)
credit, 297–9
critical
  expansion rate, 250–2
  minimum effort, theory of, 252
Culpin, C., 90(n)
curves, continuous, 8, 12, 16–17, 24–5; *see also* production function

Dean, G. W., 169(n), 172(n), 192
debt, 293; *see also* credit
decision making, 130–92; *see also* management
  tree, 146, 168
decreasing returns, 20, 88, 327; *see also* scale
demand
  for agricultural produce, 253–5
  for capital, 287–8
Denman, D. R., 220(n)
Dent, J. B., 72, 159(n)
depreciation, 262; *see also* capital
derived demand, 101
Dermott, W., 7(n)
determinants of farm growth, 331–6
Dexter, K., 75(n)
differential calculus, 28(n), 31(n), 32(n), 33(n), 35(n), 118(n), 120(n), 122(n), 125(n), 192(n)
difficulties in capital theory, 288–91
Dillon, J. L., 6(n), 35, 159(n)
diminishing
  marginal returns, 22–3, 25–6, 29–30, 35, 79, 122, 195
  marginal utility, 165
  rate of technical substitution, 18, 31, 44, 120
discount tables, 343
discounted cash flows, 271–5
discounting
  for risk, 180–2
  for time, 269–71, 310–11; *see also* time preference
discrete variable, 135–6
diseconomies, external, 100, 104; *see also* scale
disguised unemployment, 248–9
disposal
  activity, 59(n), 62
  resources in, 38, 59, 62, 111
diversification, 111
  and risk, 153–9, 180
division of labour, 89–90; *see also* specialization
Doll, J. P., 35
dominated activity, 44, 302
  strategy, 173; *see also* game theory
Dorfman, R., 72
double switching, 291, 293, 315
dynamic programming, 307–8

earnings and effort, 242, 244–5, 247
  risk–earnings possibility curve, 159–60
  transfer, 104–5
economic
  optimization, 73–129
  optimum, 94–9, 115; *see also* optimum
  optimum for a single variable input, 75–80, 117–19
  rationality, 73–5
  rent, 104(n)
economies of scale, 91; *see also* size of enterprise
  external, 100–1
  internal, 91–2
Edwards, W., 192
effective constraints, 36, 46, 66, 85, 111, 115
efficient combinations, 45; *see also* inefficient output levels
Eidman, V. R., 172(n)

elasticity of response, 8–9, 11, 20, 28–9, 32, 119, 123, 202
embodied innovations, 204
Engel's Law, 255
enterprise combinations, 21–5
enterprises, 21, 22, 107
entrepreneurship, 322–5, 331–8
environment, 5, 207, 214, 260; *see also* natural resources
equal
  advantage, principle of, 256, 258
  marginal returns, principle of, 108, 125
  probabilities, 171; *see also* principle of insufficient reason
equality of income, 230
equilibrium, 79; *see also* economic optimum
  price, 254–5
equity, 293; *see also* borrowing, capital ratio, 296–7
equivalent annuity, 282–3
Euler's Theorem, 33, 124
event, 134
expansion path, 82–5, 94, 119–21; *see also* least-cost combination
expectations, 133–8, 186–7
  –variance frontier, 159–60
    calculation by computer simulation, 159(n)
expected
  monetary value (EMV), 138, 147–8, 154–65, 168
  price, 144, 146
  revenue, 139, 143, 146
  usable yield, 148
  utility, 165–9, 296(n)
  value, 137
extending production, 208–9, 212
extensive production, 196, 207, 212
external diseconomies and economies of scale, 100–1
extreme points, 51, 59; *see also* corner points, basic feasible solution

factor
  combinations, 15–21
  demand, 99–101
  –factor relationships, 15–19
  indivisibility, 89–92
  limitations, 92–4
  –product relationships, 8, 9
  proportions, fixed, 36–9, 201, 225, 236
  substitution, 40, 202; *see also* rate of technical substitution
factors
  in disposal, 38, 59, 62, 111
  of production, 3; *see also* productive resources
fallacy of averages, 145–51, 188
family labour, 234
farm, 4
  growth, 331–6
  size, 325–9
feasible
  alternatives, 73
  area, 57, 162–3
  combination of inputs, 41
  outputs, 5, 24, 54–7
  region, 45–54
  solution, basic, 59, 62–5, 112, 127
fertility of land, 207
Fei, J. H. C., 259
Fisher, C., 6(n)
fixed
  costs, 90, 93
  -factor proportions, 36–9, 201, 225, 236
  factors, 92
  inputs, 5, 7–9, 10, 14; *see also* constraints
  -rent tenancy, 225
  resources, 21; *see also* constraints
flexibility, 182
folding back, 147–8
freehold ownership, 221, 226–9; *see also* land tenure
Friedman, M., 338
full-time labour, 233; *see also* regular labour
future value, 269; *see also* compounding of an annuity, table, 344
  table, 342

gamble, standard, 166
game theory, 172
games against nature, 172–8
gang-work days, 233, 237(n)
gearing, 296–7; *see also* equity ratio
George, H., 223(n)
Gibbs, R., 224(n)
Gibrat, R., 332(n)
Giles, A. K., 242(n), 338
Green, H. A. J., 290(n)
gross margins, 94, 96, 102–4, 106, 154–8, 281–2
  in linear programming, 113–16, 126, 149–51, 162, 238

Hall, P., 218(n)
Hallett, G., 231
Halter, A. N., 168(n), 192
Harcourt, G. C., 320
Harrison, A., 224(n), 262(n)
Hart, P. E., 338
Harvey, N., 241(n)
Hazel, P. B. R., 176(n)
Heady, E. O., 6(n), 35
heuristics, 179
Hicks, J. R., 206
hired labour, 233–4
homogeneous production functions, 32, 33(n)
hours worked, 242–7

incentives, work, 241
incoherent assignments, 170; *see also* subjective probability
income,
  average, 105, 164–5, 180; *see also* returns, revenue
  and hours worked, 242–7
  and responsibility sharing, 336
  and saving, 265, 292
  effect, 243–4
  elasticity of demand, 255(n)
  equality and land tenure, 230
  growth, and *per capita*, 252–8
  non-contractual, 323; *see also* entrepreneurship
  rent as unearned, 223
  stream of future, 204, 227, 263; *see also* present value
  survival, 162; *see also* risk
  variation, 157–8, 299; *see also* risk
increasing
  marginal product (returns), 8, 29
  rate of product transformation, 25, 58
  returns to scale, 20, 88
  risk (principle of), 296–7, 333
incremental internal rate of return, 277, 287, 331–2; *see also* internal rate of return
independent
  enterprises, 21
  events, 140, 186
individual user rights, 221, 229–30; *see also* land tenure
indivisibility,
  capital, 266, 300
  factor (input, resource), 88–91, 96; *see also* scale, fixed cost
indivisible inputs (factors, resources), 89, 99, 104, 266
induced invention, 206; *see also* technological change
inefficient activity combinations, 44–6
  cost levels, 94
  input combinations, 41
  output levels, 11
  plans, 160
  product combinations, 24
inelastic demand and supply, 254
infeasible
  activity combinations, 45
  cost levels, 95
  output levels, 11
  plans, 160
innovation and profit, 105, 323; *see also* technological change
  embodied, 204, 250
  land-saving, 202–6
  neutral, 202–6
  process, 201
input
  combinations, 15–21
  –output coefficients, 60
  –output relationships, 4–14, 20
  single variable, 8–11, 26–9, 75–80, 117–19
inputs, all variable, 20, 21; *see also* scale
  complementary, 39
  fixed, 5, 7–9, 10, 14
  indivisible, 89, 99, 104, 266
  two-variable, 11–21, 29–31, 81–4, 119–21
insufficient reason, principle of, 171, 173; *see also* subjective probability
insurance, 152, 182–3; *see also* risk
integer
  programming, 306–7
  variables, 238
intensity of land-use, 196, 203–4, 206, 212, 215–17
interaction between inputs, 7–8, 12–13, 31, 41
interest, 227, 265; *see also* capital
  compound, 268
internal
  economies of scale, 91–2; *see also* size of enterprise
  rate of return, 275–9, 314–15; *see also* capital
inverse price ratio
  factor–factor, 81–2, 87
  factor–product, 77(n), 79, 119–20
  product–product, 109–11, 126
investment, 263–7; *see also* capital

irrational areas of production, 10–11, 18, 23, 25
isocost lines, 81–3, 85–7
isolated state, von Thünen's, 218
iso-product curve, 16; *see also* production possibility curve
isoquants, 16–19, 49–52, 81, 85–7
  linear, 19, 41–5
  right-angled, 38
iso-revenue lines, 109–11
iterations, 69–70, 128–9; *see also* linear programming

Jennings, R. C., 6(n)
Johnson, G. L., 325
joint
  probability, 139, 185
  products, 21
  stock company, 337–8
Josling, T. E., 259
judgemental (personal) probability, 170

Keynes, J. M., 264(n)
King, R. A., 231
Kirzner, I. M., 320, 339

labour, 3, 196, 232–59
  force, 206
  peaks, 234–7, 240
  productivity, 240–2
  requirements, 236
  saving innovation, 201–2
  unemployment, 200, 256
Lagrange multiplier, 123(n), 125(n)
Laing, N. F., 320
Lancaster, K., 129
land, 3, 195–231
  as an effective constraint, 5, 92, 200, 210–11
  fertility (quality), 207–15
  -price gradient, 228
  saving innovation, 202–6
  tenure, 220–30
  -use zoning, 228
  values, 227–9
landlord, 220–6
landlords' capital, 222–3
law,
  Engel's, 255
  Gibrat's, 332, 334
  of diminishing returns, 6–8
  of large numbers, 134
  of proportionate effect, 332
  of variable proportions, 21
Lawson, G. H., 342
leasehold tenure, 220–4
least-cost
  activity, 86
  combination, 81–7, 94–7, 120–1
leisure–income choice, 242–7
lending, 265, 295, 304–5; *see also* credit
leverage, 296; *see also* equity
lexicographic ordering, 161; *see also* objectives
limited liability company, 337–8
limiting factors, 22, 92; *see also* constraints
limits on growth, 92–3, 329; *see also* farm size
Lin, W., 169(n)
linear
  activity, 36, 60–1
  combinations, 41
  production relationships, 36–72
  programming, 49, 65–72, 111–16, 126–9
    labour force, 237–9
    land of different qualities, 209–11
    multiperiod, 301–6
    product combinations, 111–16, 126–9
    revised simplex method, 65–72, 68(n)
    with survival constraints, 162–4
    under risk, 148–51, 176
  response, 37
linearly homogeneous production function, 32
loans, *see* lending
location and land-use, 214–20
long-run decisions, 93
lower limits/EMV curve, 164–5

McFarquhar, A. M. M., 159(n), 262(n)
MacInerney, J. P., 192
Malthus, T., 195
management, 3, 4, 14, 321–39
  decisions, 5
managerial
  ability, 5, 322, 325–9
  function, 321–2
  limits, 329–31
  pyramid, 322
man-hours (days, years), 233–47; *see also* labour
marginal
  product, 6–11, 26–9, 93
    in alternative enterprises, 22–5, 34–5
    in linear programming, 36–8, 52–4, 60, 65

marginal product *cont.*—
of alternative inputs, 11–20, 29–34, 83, 93
of capital, 274
of land, 196–200, 202–6, 208–10, 213–18, 221–2, 227
of seasonal labour, 235–6
per person, 247–9, 253
value, of the, 76–9, 118–20, 122–5, 181, 214–18, 221–2, 227
propensity to save, 292
rate of time preference, 292–5
rate of substitution, 17(n); *see also* rate of technical substitution
return on capital, 265–7, 274–5
revenue, 96, 119, 122
utility, 165
value product, 77(n); *see also* value of the marginal product
market price, 73–5, 124, 215
markets and communications, *see* location
Marris, R., 337(n)
matrix, linear programming, 62(n), 67(n), 68(n)
maximin, 165, 174–6; *see also* game theory
average product, 9–11, 26, 28
maximum earnings, 226
expected monetary value, 138, 147–8
gross margin, 96, 113–14; *see also* optimum rate of output
net present value or terminal net worth, 274–5
output, 9–11, 24, 46–54, 61–72; *see also* technical optimum
profit, 74, 109, 138; *see also* optimum
revenue, 108–11; *see also* optimum combination of products
satisfaction, 73–4
Meadows, D. H., and D. L., 195(n)
measurement
of capital, 290
of work, 232–4
mechanization, 84, 240
Metcalf, D., 75(n)
method of production, *see* factor combinations
study, 241
migration, labour, 256–8
minimax regret, 177–8; *see also* game theory
minimum
average cost, 90, 95; *see also* least cost combination
minimum *cont.*—
effective scale, 92, 96, 335
Ministry of Agriculture, Fisheries and Food, 237(n), 259
Mishan, E. J., 260(n)
mixed strategy, maximin, 174
models, 5, 6
money and capital, 3, 261
Montalbano, M., 278
Moore, C. V., 169(n)
Moore, P. G., 192
more than one activity, 39–45
Morris, T. R., 6(n)
mortgage, 298; *see also* credit
mutual exclusiveness, 40, 47, 135, 300, 306–7
multiperiod planning, 299–309, 317–19
multiple internal rates of return, 278–9; *see also* double switching

National Economic Development Office, 259
natural
environment, 207
resources, 3, 195–231
negative
capital, 278; *see also* multiple internal rates of return
growth (investment), 264, 294, 333
marginal products, 7–8, 10–11, 18, 25–6, 29
net present value, 271, 273–5, 277, 282–4, 286–8, 312–13, 315
maximum, 274–5
net worth, terminal, 268–9, 272–3; *see also* compounding
neutral innovation, 201–5
new techniques (sources of income streams), 79–80, 84, 204; *see also* innovation
normal profit, 104; *see also* transfer earnings
normative statements (theories), 74, 160, 178

objective function, 68(n), 126–7; *see also* linear programming
objectives of farmer, 74–5, 161, 165, 169, 172, 178–9, 248, 274, 301, 318–19, 327, 332
Officer, R. R., 168(n)
opportunity cost, 23, 25, 74, 100, 104, 107–11, 279, 323
of investment funds, 277, 287, 295, 299, 301, 315, 317, 319

opportunity cost *cont.*—
of land, 199, 230
optimal rate of saving and investment, 293–5
optimization over time, 279–87, 315–16
optimum,
economic,
combination of inputs (least-cost), 81–4, 120–3
combination of products, 107–11, 124–6, 214
labour force, 237–9
level of input, 76–80, 117–19, 203
rate of output, 96–9
rate of saving, 293–4
solution, 68(n), 111–16, 129, 210, 238, 301–2, 305
stock of capital, 265–7, 275
technical, 10, 46–8
orchard replacement, 284–5
outcomes,
equi-probable, 171
variable, 133–43, 147, 152–3, 168, 173
worst possible, 163–4
output, 4
effect, 101
maximum, 9–11, 24, 46–54, 61–72
overtime, 245–7
owner-occupier, 226

parallel isoquants, 43; *see also* constant returns to scale
parametric linear programming, 238
partial optimist criterion, 176–7; *see also* game theory
partnerships, 226, 336
payment by results, 241–2
pay-off, 173; *see also* game theory
peaks labour (work), 234–7, 240
pecuniary scale effects, 100
Penrose, E., 339
perfect
knowledge, 73, 130
substitutes, 18–19, 99
permanent income streams, 227, 263
perpetuities, 313–14
personal
discount rate, 292; *see also* marginal rate of time preference
(subjective) probability, 170
pledging, 220(n)
point input–point output, 267, 285–7
policy, agricultural, 258–9
population
density, 14, 196–200
growth, 206, 247–53
positive statements (theories), 74, 161
potential population growth, 252(n)
predictable price changes, 132
prediction and prescription, 74, 160
present value, 227, 269–71; *see also* discounting, net present value
of an annuity, 345
tables, 343
price
and marginal revenue, 96
cycles, 132
market, 73–5, 124, 215
movements, 84, 131
ratios, 77–9, 81–2, 109–11; *see also* inverse price ratio
trends, 132
variation, 131–3
principle
of equal advantage, 256
of increasing risk, 296–7
of insufficient reason, 171
process, 36, 85; *see also* activity (linear programming)
control operator, 232
product
combinations, 21–5
contours, 16; *see also* isoquants
exhaustion under constant returns, 33, 54, 105–7, 116
innovations, 201
price gradient, 215, 218; *see also* location
–product relationships, 21, 22, 23–5, 54–9
supply, 99–101
transformation, rate of, 23, 25, 34–5, 58, 72, 124
production, 3, 4
function, 6, 117, 124, 145, 328
period, 4, 5
-possibility boundary (curve), 24–5, 54–9, 108, 111
response curve, 3–35
productive resources, 3–4; *see also* factors of production
productivity, 9, 11, 207; *see also* average product
of labour, 237
profile, labour, 236–7
profit, 74, 105, 332–5
maximum, 74, 109, 138; *see also* optimum

profit *cont.*—
normal, 104; *see also* transfer earnings
pure, 104–5, 290
proportionate effect, law of (Gibrat's law), 332
pure
competition, 75, 131; *see also* constant prices
profit, 104–5, 290
putty-clay, 289; *see also* theoretical difficulties with capital

quadratic programming, 159(n), 191
quality of land, 207, 211

Raiffa, H., 192
Randers, J., 195(n)
random variable, 135, 186
ranking investments, 275, 277
rate of
product transformation, 23, 25, 34–5, 58, 72, 124
return, *see* return on capital
technical substitution, 17–19, 30–1, 42, 61, 81, 83, 87, 120
working, 240–2
rational behaviour, 75
Rayner, A. J., 75(n)
regret, minimax, 177–8; *see also* game theory
regular labour, 233–9
relative frequency, 133–4, 184; *see also* probability
rent, 197–9, 204, 208, 221–5, 227; *see also* land, pure profit
–distance relationship, 216–18; *see also* location
replacement problems, 279–87
resource
combinations, 15–21
indivisibility, 88–91, 96
resources,
natural, 3, 195–231
productive, 3, 4; *see also* factors of production, inputs
response
curve, 3, 9–14, 26, 78; *see also* production function
surface, 13, 16, 50
to a single variable input, 8–11
to two variable inputs, 11–14, 29
return
on capital, 261, 263; *see also* internal rate of return
to limited factors, 101–5
returns to scale, 20–1, 31–2, 84–8
revenue
and profit, 74
marginal, 96, 119, 122
maximum, 108–11
revised simplex method, 65–72
revision of probabilities, 172
Rhodes, V. J., 35
Ricardo, D., 3(n), 195, 199
ridge lines, 18, 24, 25; *see also* irrational areas
right-angled isoquants, 38
rising
cost industry, 100
marginal cost, 93–9, 119, 122
risk, 4, 5, 267, 295–9, 300
and security for loans, 298
and uncertainty, 130–92
aversion, 151–3
borrowing and, 295–9
discounting for, 180–2
diversification and, 153–9
entrepreneurship and, 323
premium, 295–6
sharing under share-tenancy, 224
Robbins, L., 92(n)
Robichek, A. A., 278(n)
Robinson, J., 260(n), 288(n), 320
rules of thumb in face of risk 178–83

saddle point, 122(n), 176
Salmon, G. D., 338
salvage value, 94, 272, 284, 289, 315–16
Samuelson, P. A., 72
satisfaction, maximum, 73–4
satisficing, 179, 307
savings, 3, 263–4, 291–5; *see also* investments
scale
coefficient, 20, 32
constant returns to, 20, 33, 54, 105–7, 124, 202, 299, 326
diseconomies of, 100–1, 104
economies of, 91–2, 100–1
increases in, 19, 36, 84, 96, 266
returns to, 19–21, 31–3
Schmidt, S., 320
seasonal
labour requirements, 234–40
unemployment, 235
security
collateral, 298; *see also* credit
in land ownership, 229
of tenure, 222–3, 226

shadow-prices, 68(n), 103, 106, 115–16, 123–4; *see also* opportunity cost
share-tenancy, 220, 224–6
sharing entrepreneurial responsibility, 336–8
short-run decisions, 93
simplex method, revised, 65–72
simulation, computer, to calculate EMVs and variances, 159(n), 309
single variable input, response to, 8–11
size distribution
  of agricultural sector, 253–9
  of enterprise (farm), 325–9
  of farms, 334–6
slack variable, 62; *see also* disposal activity
Smith, A., 195, 196(n)
social overhead capital, 260
soils and land-use, 207–14
Solow, R. M., 72
spacing of isoquants, 19
specialization, 89–90, 111, 213–14, 219, 330
specific resources, 21, 22
speculation and land values, 228, 229
Spivey, W. A., 72
stable income, preference for, 299–300
stages of response, 10, 11
standard
  deviation, 153
  gamble, 166
  man-hours, 233
State as landlord, 224
states of nature, 173, 224
steady-state equilibrium, 334
Steindl, J., 339
straight-line
  expansion path, 84–7; *see also* fixed factor proportions
  isoquants, 19
  relationships, 21, 36, 85; *see also* linear production relationships
strategy, 172; *see also* game theory
sub-marginal land, 211–12, 217–18
subjective
  probability, 169–72, 178
  risk, 171
subsistence
  income necessary for, and leisure, 242–4
  needs and disguised unemployment, 248
substitutes, perfect, 18–19; *see also* factor–factor relationships
substitution
  activity, 40–9
  ease of, 44
  effect, 101, 243–4
supplementary enterprises, 25, 59, 111, 236
supply
  and demand for agricultural produce, 253–5
  and demand for new capital, 331–4
  curve of effort, 244
  function, 99
  price for entrepreneurship, 324
surface, response, 12–13, 16–17, 31, 50
survival needs, 73, 161–3

tax on rents, 223
technical
  efficiency, 207
  optimum, 10, 46
  relationship, 5, 73
  substitution, rate of, 17–19, 30–1, 42, 61, 81, 83, 87
techniques, new, 79–80, 324
technological change, 5, 201–7
Teichrow, D., 278(n)
terminal
  assets, 301–6, 318–19; *see also* multi-period planning
  cost, 215; *see also* transfer cost
  net worth, 268–9, 272–3; *see also* compounding
  optimality, 319
terms of trade for agriculture, 255(n)
Thornton, D. S., 259
Throsby, C. D., 72
Thünen, J. H. von., 218–19
time preference, marginal rate of, 292, 304
total
  cost, 74, 76–80
  product, 6–11, 22, 25, 26
  revenue, 108–10, 119
tractor replacement, 285–6
trading, 213–14; *see also* comparative advantage
transfer
  cost, 215–16; *see also* location
  earnings, 104–5; *see also* opportunity cost
transport cost, 214–16; *see also* location
tree diagrams, 141–4
Tuck, R. H., 259, 339
Tversky, A., 192
two variable factors (inputs, resources), 11–19, 29–31, 81–4, 119–21

uncertainty, 73, 169–72; *see also* risk
unearned incomes, rents as, 223
unemployment, 200, 256–7
  disguised, 248–9, 252
unit factor cost, 76–9, 97, 118, 122
unsecured credit, 298
unused capacity, 91
Upton, M., 234(n), 242(n)
utility, expected, 165–9

value
  of capital, 290
  of land, 227–9
  of the marginal product, 76–9, 118–20, 122–5, 181, 214–18, 221–2, 227
Vandermeulen, D. C., 72
variable costs, 95–6, 100
  inputs, single, 8–11, 26–9, 75–80, 117–19
  inputs, all, 20, 21; *see also* scale
  inputs, two, 11–21, 29–31, 81–4, 119–21
variance
  and covariance, 152–60, 168, 188–92
  calculation by computer simulation, 159(n)
variation in agricultural production, 5, 130–1
vertical integration, 183, 336

wage rate, 232–4, 241–4, 246–7; *see also* labour
Weingartner, M., 320
West, J. G., 35
Wicksell effect, 290(n)
Windle, D. W., 342
Worcester, D. A., 231
work
  measurement, 232–4
  peaks, 234–7, 240; *see also* seasonality
  study, 241
working capital, 224, 262

yield, 6, 15, 207
  of investment, 275; *see also* internal rate of return

zero
  marginal product, 9, 18, 26, 29, 103, 115; *see also* technical optimum
  profit equilibrium, 105